Fermilab

TRANSPORT APPENDIX

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APPENDIX

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APPENDIX

Introduction

This appendix has been included as an addition to the manual in an attempt to better acquaint the user with 'what TRANSPORT does', and with the notation and mathematical formalism used in a TRANSPORT calculation.

The first section (Beam Transport Optics - Part I and Part II) is a rewrite of two lectures given to members of the SLAC technical staff on the elementary matrix algebra of optics. We include them here for the benefit of the new user who may need a brief refresher course on charged particle optics and/or has a need to become familiar with TRANSPORT notation. The new user should also acquaint himself with the contents of the books and other publications listed under 'references' at the end of the manual. References 1 and 2 are essential if the user is to obtain the maximum value from TRANSPORT.

The second section of this appendix was written to introduce the mathematical formalism of the first-order R matrix and Sigma matrix (phase ellipsoid) beam optics used in a TRANSPORT calculation and to correlate this with the printed output.

Section three discusses second-order calculations and, in particular, a procedure for calculating the "Sextupole" strengths required to minimize and/or eliminate second-order aberrations in a beam transport system.

Section four is a brief derivation of the mathematical formalism used by TRANSPORT for calculating magnet alignment tolerances.

Section five deals with the first-order parameter optimization code of TRANSPORT and includes a brief explanation of the covariance matrix that is printed after each first-order fit routine.

BEAM TRANSPORT OPTICS

Section I

Beam Transport Optics - Part I

(K. L. Brown)

1. Introduction

A convenient starting point for this lecture is the equation relating the magnetic rigidity of a particle $(B\rho)$ to the particle momentum P .

$$B\rho = \frac{10^2}{2.99793} P \quad \text{or} \quad B\rho = 33.356\ P$$

where

B is in kilogauss

ρ is the bending radius in meters

P is the particle's momentum in BeV/c.

A note of caution: When using this equation for a TRANSPORT calculation, it is necessary to use at least 5 significant figures for the constant to avoid round-off errors in the readout.

2. Geometric Light Optics vs. Magnetic Optics

To relate geometrical light optics to charged particle optics, we begin with the thin lens. Figure 1 shows a thin lens with a ray leaving a focal point A at an angle θ_o , impinging on the lens at x_o . As the ray leaves the lens, it is at x_1 and going toward a focal point B at an angle of θ_1 .

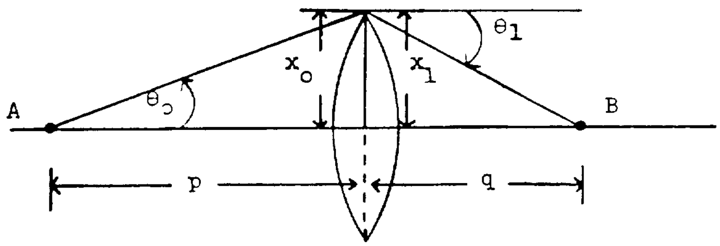


Figure 1

Thin lens optics says that $1/p + 1/q = 1/f$. Using this equation it is readily verified that the matrix transformation for the lens action between principal planes is

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_o \\ \theta_o \end{bmatrix}$$

The transformation for a drift distance L is

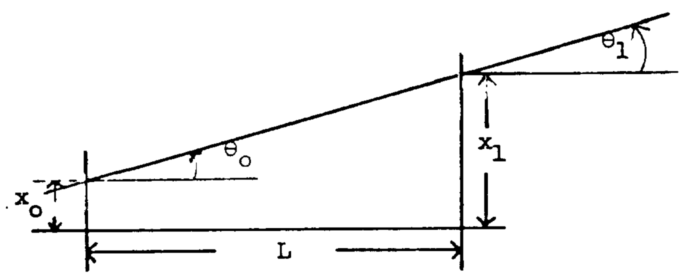


Figure 2

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ \theta_o \end{bmatrix}$$

Note that the determinant of the matrix in both examples is equal to unity. This is always the case as will be proved formally later. That this is so is a manifestation of Liouville's theorem of conservation of phase space area.

Consider now a thick lens, as illustrated in Figure 3.

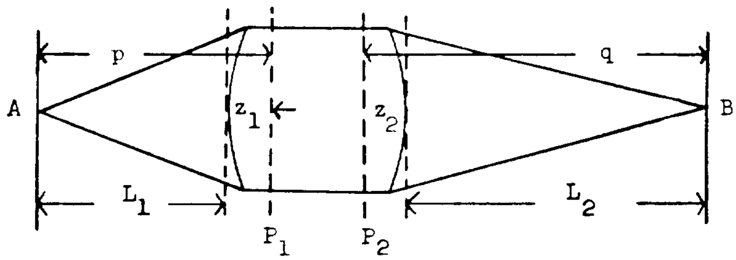


Figure 3

If L_1 is the object distance to the face of the lens and L_2 is the corresponding image distance, then, in general,

$$1/L_1 + 1/L_2 \neq 1/f \quad .$$

If, however, we introduce two planes P_1 and P_2 located at a distance z_1 and z_2 from the entrance and exit faces of the lens, it is always possible to find a z_1 and a z_2 such that the equation

$$1/p + 1/q = 1/f \quad \text{is valid.}$$

where

$$p = L_1 + z_1$$
$$q = L_2 + z_2 \quad .$$

When this is so, P_1 and P_2 are called the principal planes of the lens.

Now, relating the above statement to matrix formalism, the matrix transformation for a thick lens between the input and output faces of the lens has the general form:

$$\begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x_o \\ \theta_o \end{bmatrix} \qquad (1)$$

where as before the det $R = 1$. For a general transformation, R_{12} is not necessarily equal to 0 and R_{11} and R_{22} are not necessarily equal to 1 .

The principal planes may be located by the transformation

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix} \qquad (2)$$

Using the relation

$$\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \text{ (the unit matrix)}$$

the previous equation may be manipulated into the form

$$\begin{bmatrix} R_{11}-z_2R_{21} & R_{12}-z_1R_{11} -z_2(R_{22}-z_1R_{21}) \\ R_{21} & R_{22}-z_1R_{21} \end{bmatrix} = \begin{bmatrix} 1 & -z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 1 & -z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \qquad (3)$$

Solving for z_1 and z_2 , we find

$$z_1 = \frac{R_{22} - 1}{R_{21}}$$

$$z_2 = \frac{R_{11} - 1}{R_{21}}$$

where z_1 and z_2 are the location of the principal planes as shown in Fig. 3. The principal planes of <u>any</u> system may be determined by this method.

Note that $R_{21} = -1/f$ is not affected by the transformation and that the upper right hand matrix element is zero if det R = 1 . The principal planes may coincide, may be close together, be far apart; or in many systems, may be located external to all of the elements comprising the system. An example of the latter case is a quadrupole pair.

Some examples of principal plane locations for simple systems follow:

A quadrupole singlet:

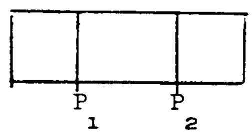


Figure 4

The principal planes in a single quadrupole are located very close to each other and very near the center of the lens. As such, a quadrupole singlet may be considered as a thin lens if the object and image distances are measured to the center of the lens.

A simple uniform-field wedge magnet:

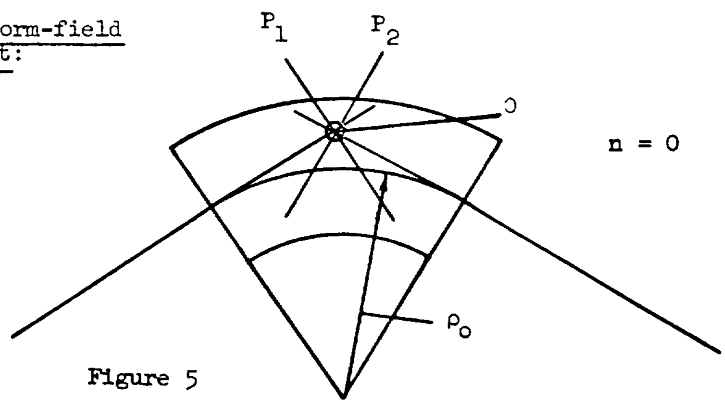
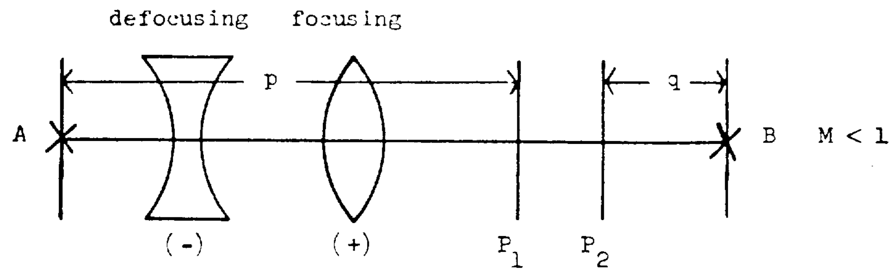


Figure 5

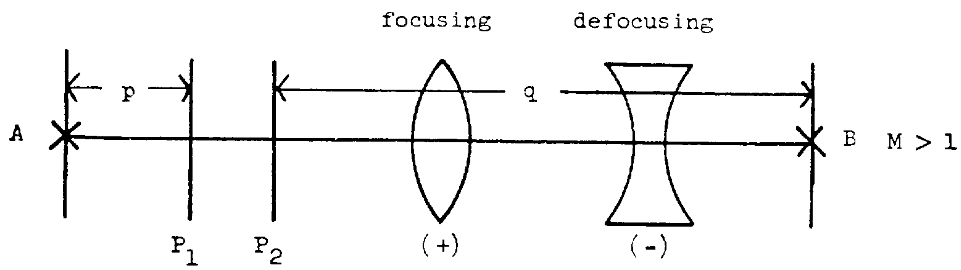
If the optic axis enters and exits perpendicularly to the pole boundary, the principal planes are at the "center" of the magnet, as shown in Figure 5. From this, we conclude that a simple wedge bending magnet may be considered as a "thin" lens if the object and image distances are measured to the lens center O .

A quadrupole pair:



x plane

Figure 6



y plane

Figure 7

For a quadrupole pair, the principal planes are displaced toward and, usually, beyond the focusing element of the pair, as shown in Figures 6 and 7.

For any lens system, no matter how many elements are involved

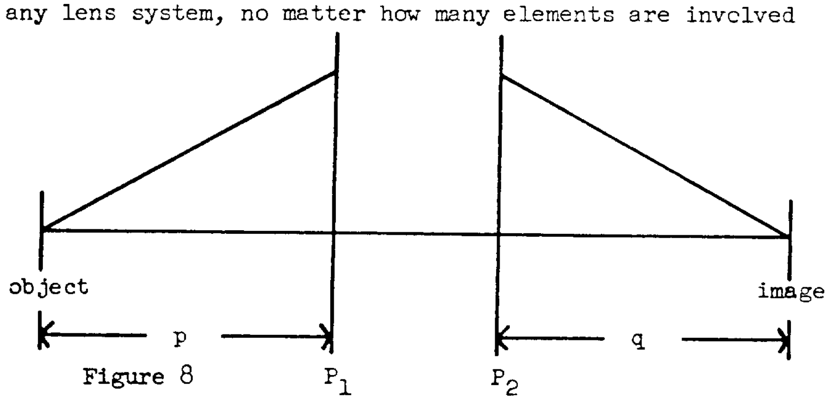


Figure 8

$1/p + 1/q = 1/f$, if p and q are distances measured to the principal planes. Then the magnification between object and image planes is $M = q/p$.

Since the quadrupole pair is different in the two planes, (x) and (y), both situations must be examined. The interesting result turns out to be that in the x plane, the principal planes are to the right (Figure 6) and in the y plane, they are to the left (Figure 7). Therefore, in the y plane the magnification is greater than 1, and in the x plane the magnification is less than 1. Typically, for a quadrupole pair the ratio of

$$M_y/M_x$$

may be as high as 20:1 and such cases can be disastrous if not recognized beforehand. This is a first-order image distortion. For example, if the source is a circular spot at A , the image at B will appear as a long thin line.

The situation is different for the

Quadrupole triplet:

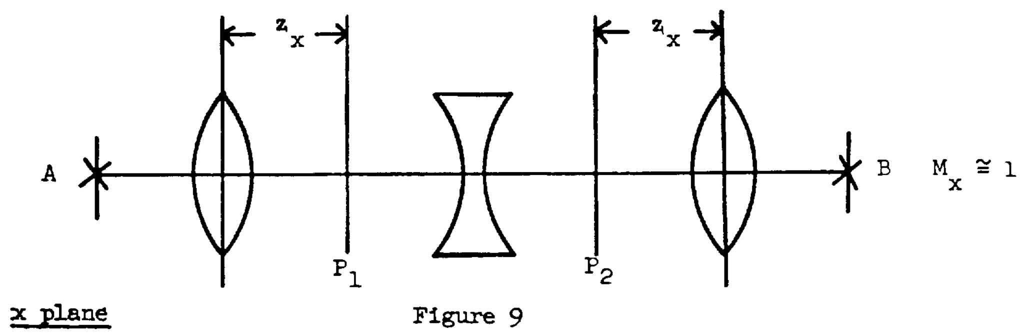


Figure 9

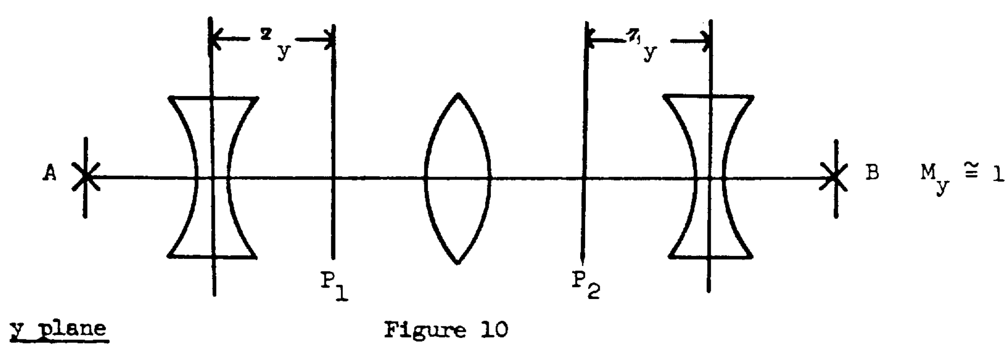


Figure 10

In the symmetric triplet, as shown in Figures 9 and 10, the principal planes are located symmetrically about the center of the system, although $z_x > z_y$. This is, perhaps, the dominant reason why quadrupole triplets are used. The magnification is approximately equal in both planes; consequently, a circular spot can be imaged through the system with much less first-order image distortion than is the case for the doublet.

3. Introduction of Momentum Dispersion into the Matrix Formalism

The foregoing discussion and examples dealt only with monoenergetic first-order effects. First-order dispersion may be taken into account by introducing a 3 × 3 matrix as follows:

Consider two particles of momentum p_o and $p_o + \Delta p$ passing through the midplane of a static magnetic field, as illustrated in Figure 11.

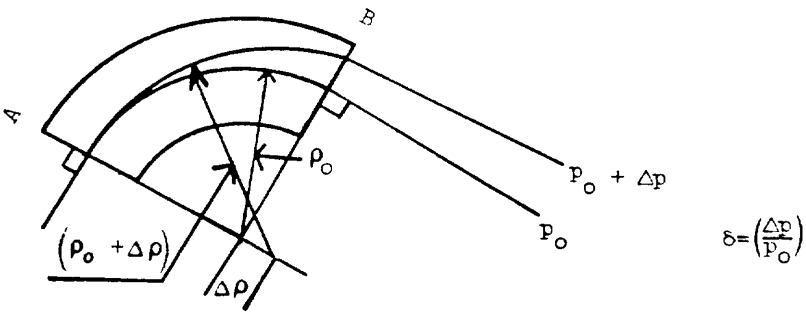


Figure 11

Since the scalar momentum of a particle is constant in a static magnetic field, the transport equation from A to B may be expressed as:

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix}_B = \begin{bmatrix} R_{11} & R_{12} & d \\ R_{21} & R_{22} & d' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix}_A$$

where

$\delta = \Delta p/p_o$

d = the spatial momentum dispersion

d' = the derivative of the dispersion (the angular momentum dispersion)

and 1 = a carrying term to generate a square matrix and denote a constant momentum.

The determinant of the matrix $|R|$ is equal to 1 as for the 2 × 2 matrix. However, because of the zeros in the bottom row, the fact that $|R| = (R_{11} R_{22} - R_{12} R_{21}) = 1$ only checks the 2 × 2 matrix and not the terms containing d and d' .

Consider now a general system from an object point A to an image point B.

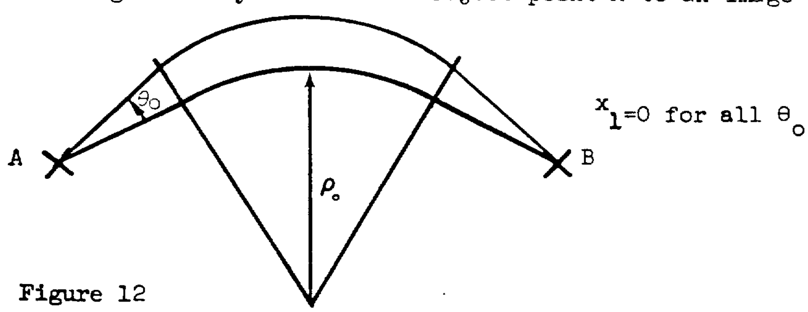


Figure 12

The above matrix equation is still valid for midplane trajectories. If A is a source point and if $R_{12} = 0$ (i.e., x is independent of θ_0), then B is an image point for monoenergetic particles. Under these circumstances:

$$R_{11} = M_x \text{ is the x plane magnification}$$

$$R_{22} = + 1/M_x \text{ because det } R = 1$$

and $$R_{21} = - 1/f_x$$

In fact, $R_{21} = - 1/f_x$ for the system between A and B , even if A and B are not foci.

It is now convenient to develop a more general definition of the matrix elements R_{ij} and, at the same time, introduce the first-order matrix transformation for the y(non-bend) plane. Consider, again, a general system where the projection of the central trajectory is allowed to bend in the x plane but is a straight line in the y plane. The x plane and y plane matrix transformations may be written as follows:

For the x plane $x_1 = R_x\, x_o$

or

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = \left[\begin{array}{c|c|c} c_x(t) & s_x(t) & d_x(t) \\ \hline c_x'(t) & s_x'(t) & d_x'(t) \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x_o \\ \theta_o \\ \delta \end{bmatrix}$$

Similarly, for the y plane, $y_1 = R_y\, y_o$

or

$$\begin{bmatrix} y_1 \\ \Phi_1 \end{bmatrix} = \left[\begin{array}{c|c} c_y(t) & s_y(t) \\ \hline c_y'(t) & s_y'(t) \end{array}\right] \begin{bmatrix} y_o \\ \Phi_o \end{bmatrix}$$

The c and s functions may be defined in terms of their initial conditions. Let τ be the distance measured along the

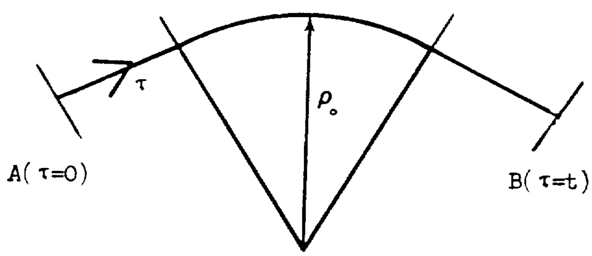


Figure 13

central trajectory. Then:

$s(0) = 0 \qquad s'(0) = 1 \qquad$ where $\qquad s'(\tau) = \frac{ds(\tau)}{d\tau}$

$c(0) = 1 \qquad c'(0) = 0 \qquad\qquad c'(\tau) = \frac{dc(\tau)}{d\tau} = -\frac{1}{f}$

Within an "ideal" magnet, where the bending radius ρ_o is constant, s and c are sine and cosine or else sinh and cosh functions. Because of this, the terminology s = a sine-like function and c = a cosine-like function has been adopted for describing the general case where $\rho_0 = \rho_0(\tau)$ is a function of τ.

By analogy with previous discussions, we observe that whenever $s(\tau) = 0$, we are at an image of point A. Also, $c(\tau)$ at the position where $s(\tau) = 0$ is the magnification of point A at that image.

$c'(t) = -1/f$ where f is the focal length of the system between A and B. The dispersion d_x may be derived from the general differential equations of motion of a charged particle in a static magnetic field.[1] The results may always be expressed as a function of s_x and c_x as follows:

$$d_x(t) = s_x(t) \int_0^t c_x(\tau)d\alpha - c_x(t) \int_0^t s_x(\tau)d\alpha$$

and

$$d_x'(t) = s_x'(t) \int_0^t c_x(\tau)d\alpha - c_x'(t) \int_0^t s_x(\tau)d\alpha$$

where

$$d\alpha = \frac{d\tau}{\rho_0(\tau)}$$

is the differential angle of bend of the central trajectory. At an image point $[s(t) = 0]$ note that

$$d_x(t) = -c(t) \int_0^t s(\tau)d\alpha$$

This approach to the problem may be generalized to include all of the second-order aberrations of a system. When this is done, it is always

(1) See SLAC-75 for a derivation of these equations.

possible to express these aberrations as functions of the first-order matrix elements c_x, s_x, d_x, c_y and s_y .

Having developed the above physical concepts and mathematical tools, we are now in a position to study more complicated systems. As an example, we consider the general system shown in Figure 14.

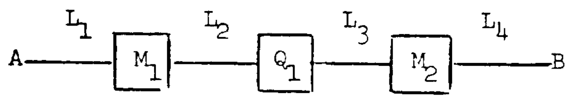


Figure 14

L = drift elements

□ = magnetic elements

The matrix formalism states that in the x plane, the transformation from A to B is given by the following matrix equation.

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = R_{L_4} R_2 R_{L_3} R_{Q_1} R_{L_2} R_1 R_{L_1} \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \end{bmatrix}$$

As in all matrix calculations, the order of writing down the elements comprising the system is from right to left. The individual matrix elements must be derived from the solution of the equation of motion within each element. If this has been done, then the calculation for the total system is carried out in the fashion shown by the above equation.

4. Second-Order Matrix Formalism (1)

It is possible to extend the 3×3 matrix formalism to solve simultaneous sets of power series by generating a second-order matrix equation as follows:

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \\ x_1^2 \\ x_1\theta_1 \\ x_1\delta \\ \vdots \end{bmatrix} = \left[\begin{array}{c|c} \begin{array}{c} 3 \times 3 \\ \text{first-order terms} \\ \text{"}R_1\text{"} \end{array} & \begin{array}{c} \text{second-order} \\ \text{terms} \end{array} \\ \hline \text{all zero} & \text{"}R_1^2\text{"} \end{array}\right] \begin{bmatrix} x_0 \\ \theta_0 \\ \delta \\ x_0^2 \\ x_0\theta_0 \\ x_0\delta \\ \vdots \end{bmatrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{all other order terms}$$

The "R_1^2" term is obtained by squaring the upper left corner (3 × 3) matrix so as to obtain second-order equations for x_1^2 , $x_1\theta_1$, $x_1\delta$, etc., as functions of products of the initial first-order variables x_0, θ_0, and δ .

This is, then, a convenient mathematical formalism for keeping all the terms desired and dropping those undesired. In the above example, all first- and second-order terms are retained and all higher-order terms are automatically dropped by the matrix multiplication.

5. Transformations Involving Many Trajectories

All of the discussion to this point relates to the transformation of a single trajectory (in addition to the central trajectory) through a static magnetic system. We wish now to extend the discussion to include "bundles" of rays. To accomplish this, we take advantage of Liouville's Theorem, which states that the "phase space" is conserved through the system. While the Theorem is strictly true to all orders, a convenient mathematical transformation has only been developed to first-order. A manifestation of Liouville's Theorem is the fact that $|R| = 1$.

Now, so long as there is no coupling mechanism between the x plane and y plane of a magnetic-optical system (which is the case if the midplane symmetry prevails throughout the system) then, the phase space area in a given plane is also conserved. Consider a bundle of rays represented by the parallelogram, shown in Figure 15(a), representing the phase space distribution of the rays at some initial position. If we now look at the phase space distribution of the same bundle after it has drifted down stream, we observe the the θ_{max} boundary and the x intercept x_i remain unchanged. In other words, the area of the parallelogram is the same or "phase space area has been conserved."

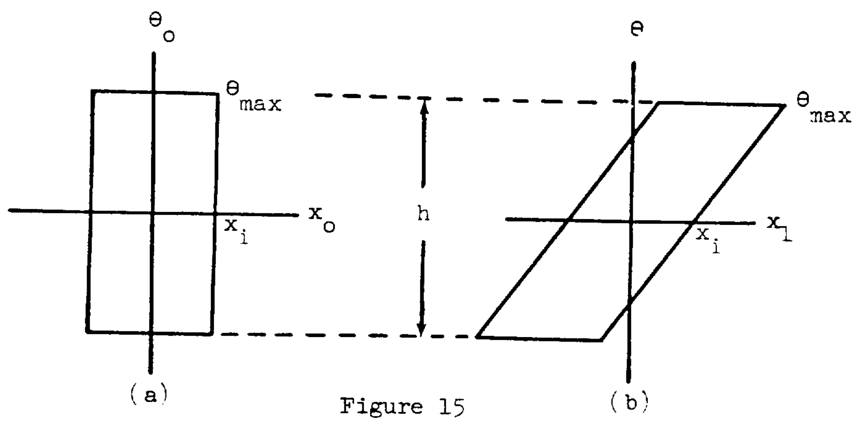


Figure 15

For mathematical convenience, the parallelogram is rather difficult to work with and, hence, a phase ellipse is usually used.

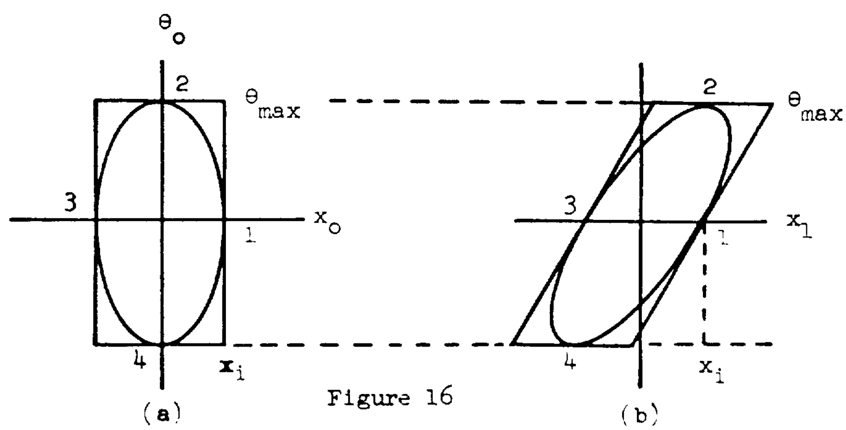


Figure 16

The phase ellipse transformation for a drift distance is illustrated in Figure 16. Figure 16(a) corresponds to a beam which is at its minimum width (a "waist") and Figure 16(b) shows the same beam after it has drifted downstream from the waist position. The physical meaning of this is that particles entering at $\theta = 0$ are parallel to the optic axis and, therefore, cannot change their relative positions with respect to the optic axis; that is, all particles on the x_o axis act in this manner. Those that enter at a given angle continue at the same angle.

The phase ellipse transformation for a thin lens is illustrated in Figure 17. In passing through a thin lens, θ changes and the x dimension remains constant for a given trajectory.

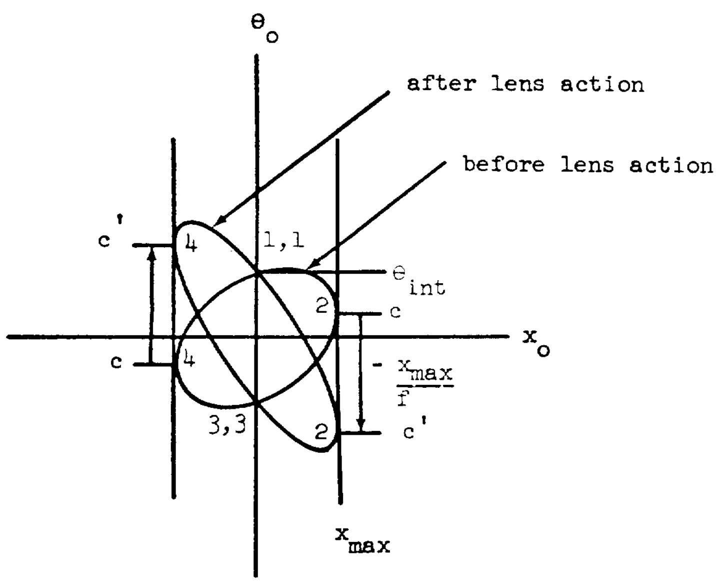


Figure 17

Stated in other terms, x_{max} remains constant, as does θ_i (the θ intercept). It is apparent, in the given example, that the spot size now becomes, or can become, smaller at the new image because θ_{max} is larger. This can be related to the physics of the system by saying that the x magnification is less than unity. This fact is observed directly by comparison of the x intercept of the ellipse before and after the lens action. It is interesting to observe that a particle initially at c is transformed to c' and that particles entering at $x = 0$ do not change their direction (θ_i is constant). If the particles are now allowed to drift, the ellipse rotates clockwise; when the ellipse if vertical, the spot size is at a minimum, namely, $x_{max} = x_i$, as was illustrated in Figure 16.

Beam Transport Optics - Part II
(K. L. Brown)

1. Introduction

In Part I, the basic concepts of beam transport optics were established. Starting from the essentials of geometric optics, the methods of matrix algebra were introduced with the example of calculating the principal planes of a thick lens. The 3 × 3 matrix for the first- order beam transport calculations were introduced to take into account the particle momenta.

2. First Order Transformation Matrix

Figure 1 shows a general region containing a magnetic field.

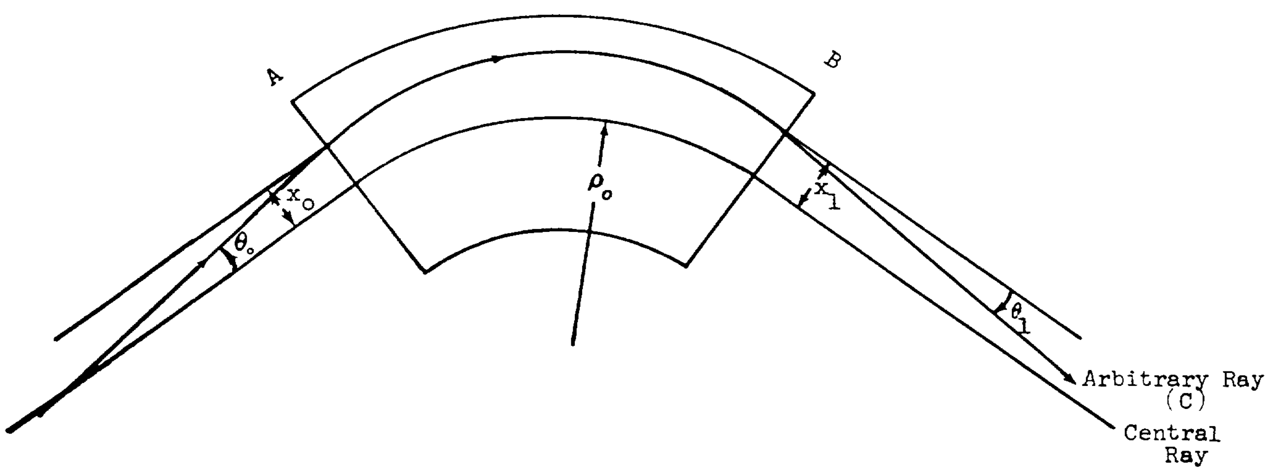


Figure 1: General Magnetic Field Configuration

The matrix presents a convenient way of writing the family of equations which describe the transformation from surface A to surface B. If x_o, θ_o and δ represent the conditions of a ray entering the system at A, then the conditions of the ray at B are x_1, θ_1 and δ. Here x_o is the distance from the central ray to the ray C, Θ_o is the angle between C and the parallel to the central ray and δ is the ratio $\Delta p/p$ where Δp is the difference between the momentum of C and the momentum of the central ray.

The linear transformation equations are:

$$
\begin{aligned}
x_1 &= c_x x_o + s_x \theta_o + d_x \delta \\
\theta_1 &= c'_x x_o + s'_x \theta_o + d'_x \delta \\
\delta &= 0 + 0 + \delta
\end{aligned}
\tag{1}
$$

Expressed as a matrix, Eq. (1) are:

$$
\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = \begin{bmatrix} c_x & s_x & d_x \\ c'_x & s'_x & d'_x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ \theta_o \\ \delta \end{bmatrix} \tag{2}
$$

The equation $\delta = \delta$ expresses the fact that the magnetic field cannot change the scalar momentum of the particle. The δ terms in the x and θ equations express the momentum dispersion of the system.

If it happens that A is an object point and B is an image point of the system, then x_1 is independent of θ_o, thus $s_x = 0$. In this case, c_x is given by $c_x = x_1/x_o = M_x$ = the magnification in x plane, (for $\delta = 0$). if $\theta_o = \delta = 0$, then $\theta_1 = c'_x\, x_o = - x_o/f$ or $c'_x = - 1/f$. It must always be true that the determinant of the matrix $|R|$, is unity. Thus for this special case of $s_x = 0$, it follows that $s'_x = 1/M_x$.

3. Beam Switchyard

As an example of a system which can be calculated with the matrix method, we next consider the beam switchyard of the two-mile accelerator. Figure 2 shows the three essential elements, two bending magnets and a quadrupole lens. In common with many beam transport systems, this one is designed to be achromatic. Mathematically, this means that the matrix elements, d_x and d'_x should be zero, so that there is no x or θ dependence on the momentum of the particles.

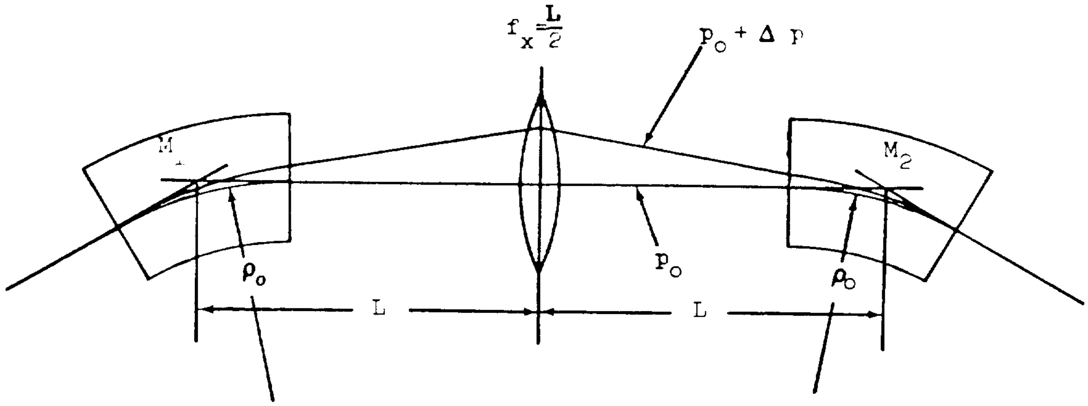


Figure 2: The Essentials of the Beam Switchyard

As a preliminary step, we will find the matrix expression for a bending magnet when measured from the principal planes. The matrix for a bending magnet, when measured from the ends of the poles, is given by:

$$R_{bend} = \begin{bmatrix} c & s & 1-c \\ -s & c & s \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

where $c = \cos \alpha$ and $s = \sin \alpha$ and α is the deflection angle of the central ray. This expression has been normalized by setting the bending radius equal to unity. To restore ordinary units it is only necessary to insert the bending radius wherever a length is needed dimensionally. In this case, the matrix then becomes

$$R_{bend} = \begin{bmatrix} c & \rho s & \rho(1-c) \\ -s/\rho & c & s \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

If the distance from the entrance plane to the first principal plane is z_1 and the distance from the second principal plane to the exit plane is z_2, we can find the values z_1 and z_2 by solving the following matrix equation:

$$\begin{bmatrix} 1 & -z_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & s & 1-c \\ -s & c & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ -1/f & 1 & d'_x \\ 0 & 0 & 1 \end{bmatrix} \qquad (5)$$

The matrix multiplication need only be done for the 2 × 2 matrices as outlined. To illustrate matrix multiplication the indicated operations will be given below in natural stages as follows:

$$\begin{bmatrix} 1 & -z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & -cz_1 + s \\ -s & +sz_1 + c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

$$\begin{bmatrix} c + z_2 s & -cz_1 + s + sz_1 z_2 - cz_2 \\ -s & +sz_1 + c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \qquad (6)$$

Note that these transformations do not change the focal length expression, $-1/f = -s$. In order for two matrices to be equal, each individual element must be equal to its counterpart in the other matrix. Thus we have

$$\begin{aligned} c + z_2 s &= 1 \\ c + z_1 s &= 1 \end{aligned} \qquad (7)$$

which when solved for z_1 and z_2 yield

$$z_2 = (1-c)/s \quad \text{and} \quad z_1 = (1-c)/s \quad .$$

If we substitute the trigonometric equivalents, and apply standard identities, we have $z_2 = z_1 = \tan(\alpha/2)$ which can be seen from Figure 3 to indicate that the two principal planes are coincident with the symmetry plane in the middle of the magnet.

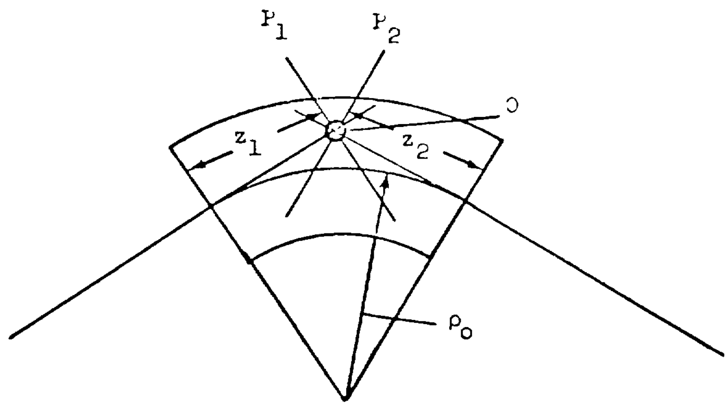


Figure 3: The Principal Planes of a Simple Bending Magnet are Coincident with the Center Plane

The simplified matrix for a bending magnet measured to the principal planes is then:

$$R_{bend} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -s & 1 & s \\ \hline 0 & 0 & 1 \end{array}\right] \tag{8}$$

To calculate the transformation matrix for the entire Beam Switchyard system as shown in Fig. 2, we write the matrices in opposite order from that in which the beam passes through the elements. That this must be true can be seen from the way in which one element alone is calculated by

$$\begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = R_1 \begin{bmatrix} x_o \\ \theta_o \\ \delta \end{bmatrix} \tag{9}$$

Then for a second element we have

$$\begin{bmatrix} x_2 \\ \theta_2 \\ \delta \end{bmatrix} = R_2 \begin{bmatrix} x_1 \\ \theta_1 \\ \delta \end{bmatrix} = R_2 R_1 \begin{bmatrix} x_o \\ \theta_o \\ \delta \end{bmatrix} \tag{10}$$

and so forth.

If we allow the system to be symmetrical, i.e. $s_1 = s_2$ and $L_1 = L_2$; the complete series of matrices for Fig. 2 are

$$R_{BSY} = \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \qquad (11)$$

We will show the step-by-step multiplication of the matrices to get the d_x and d_x' terms.

$$R_{BSY} = \begin{bmatrix} 1 & L & 0 \\ -s & 1-sL & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-sL & L & Ls \\ -s & 1 & s \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{BSY} = \begin{bmatrix} (1-\frac{L}{f}) - Ls(2-\frac{L}{f}) & L(2-\frac{L}{f}) & Ls(2-\frac{L}{f}) \\ (Ls-1)\left[2s+\frac{1}{f}(1-Ls)\right] & (1-\frac{L}{f}) - Ls(2-\frac{L}{f}) & s(1-Ls)\ (2-\frac{L}{f}) \\ 0 & 0 & 1 \end{bmatrix} \qquad (12)$$

To obtain the required condition that $d_x = d_x' = 0$, we set

$$\left(2 - \frac{L}{f}\right) = 0 \quad \text{or} \quad f = \frac{L}{2}$$

then

$$R_{BSY} = \begin{bmatrix} -1 & 0 & 0 \\ -\frac{2}{L}(1-Ls) & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (13)$$

Thus the quadrupole acts as a lens to refocus rays from the center of the first bending magnet to the center of the last one. For the serious student, it is a worthwhile exercise to do the BSY problem without the simplification which resulted from introducing the principal planes.

4. TRANSPORT

As an aid to solving beam transport problems, a computer program TRANSPORT has been developed at SLAC which takes the greatest amount of labor out of this work. The program operates in about the way as the BSY example above was calculated, but with some important exceptions. Most importantly:

1. TRANSPORT has the ability to find the best first-order solution given a certain set of constraints;

2. TRANSPORT also calculates the transformation of a whole family of rays as found in a beam by means of the concept of "phase space" which was introduced in Part I;

3. TRANSPORT can, as an option, calculate the second order effects on the beam. By second order is meant, for example, terms which depend not linearly on the displacement x_o , but on x_o^2 or $x_o \theta_o$, etc.

To aid in the discussion of TRANSPORT and of the second order terms we now introduce an abbreviated notation. By writing out the complete equations for x and y , to second order, we will adequately have displayed the new notation.

$$\begin{aligned} x_1 = {} & (x|x_o)x_o + (x|\theta_o)\theta_o + (x|\delta)\delta \\ & +(x|x_o^2)x_o^2 + (x|x_o\theta_o)x_o\theta_o + (x|x_o\delta)x_o\delta \\ & +(x|\theta_o^2)\theta_o^2 + (x|\theta_o\delta)\theta_o\delta + (x|\delta^2)\delta^2 \\ & +(x|y_o^2)y_o^2 + (x|y_o\Phi_o)y_o\Phi_o + (x|\Phi_o^2)\Phi_o^2 \end{aligned} \qquad (14)$$

$$\begin{aligned} y_1 = {} & (y|y_o) + (y|\Phi_o)\Phi_o \\ & +(y|x_oy_o)x_oy_o + (y|x_o\Phi_o)x_o\Phi_o + (y|\theta_oy_o)\theta_oy_o \\ & +(y|\theta_o\Phi_o)\theta_o\Phi_o + (y|y_o\delta)y_o\delta + (y|\delta\Phi_o)\delta\Phi_o \end{aligned} \qquad (15)$$

The absence of certain terms which might otherwise be expected in Eqs. (14) and (15) is due to the fact that horizontal mid-plane symmetry has been assumed in the derivation. That is, the field on the horizontal mid-plane is normal to the plane. Thus there can be no $(y|x)$ or $(y|\theta)$ term. Similarly, there can only be even powers of y and Φ, such as $(x|y_o^2)$ and $(x|y_o\Phi_o)$, in the x equation. Also, note that there is no $(y|\delta)$ or $(y|\delta^2)$ term if there is mid-plane symmetry.

TRANSPORT uses a numerical notation to signify the six basic coordinates:

$$\begin{array}{cccccc} x & \theta & y & \Phi & \ell & \delta \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \qquad (16)$$

The ℓ term has not been introduced here before. Its significance is the preservation of the bunch length of a beam such as the SLAC electron beam.

The first order output from TRANSPORT is a 6 × 6 matrix printout of the R matrix where the labels are implied by row and column position of the elements. For example the element appearing at the intersection of row 3 and column 4 is the coefficient $(y|\Phi_o)$ etc.

	x_o	θ_o	y_o	Φ_o	ℓ_o	δ_o
x	-	-	-	-	-	-
θ	-	-	-	-	-	-
y	-	-	-	-	-	-
Φ	-	-	-	-	-	-
ℓ	-	-	-	-	-	-
δ	-	-	-	-	-	-

The second order terms are labelled by the convention indicated in Eqs. (14), (15) and (16). For example, $(x|x_o^2)$ becomes 1 11 and $(x|\theta_o\delta)$ becomes 1 26.

5. Second Order Matrix

Normally the matrix method is expected only to apply to the solution of linear, i.e. first order, equations. However, the method has been extended to include second order terms as discussed in Part I.

For a more extensive discussion of the second-order matrix formalism, the reader is referred to SLAC report number 75 by K.L. Brown.