SECOND-ORDER ABERRATIONS

Section III

TRANSPORT has the capability of calculating the second-order matrix elements (aberration coefficients) of any static-magnetic beam transport system composed of combinations of bending magnets, quadrupoles, solenoids, sextupoles and interspersed drift spaces. It is assumed that mid-plane symmetry prevails for any given magnetic element in a system (except for solenoids) but not necessarily for the system as a whole. The notation used in a TRANSPORT printout is described in reference 1 (SLAC-75) beginning on page 46. The subscript notation is the same as that used for first-order where the subscript 1 means x, 2 means θ, 3 means y, 4 means φ, 5 means ℓ, and 6 means δ.

The symbol R_{ij} has been used to signify a first-order matrix element and the symbol T_{ijk} will be used to signify a second-order matrix element. Thus we may write the second-order Taylor expansion representing the deviation of an arbitrary trajectory from the central trajectory as:

$$x_i(t) = \sum_{j=1}^{6} R_{ij}x_j(0) + \sum_{j=1}^{6}\sum_{k=1}^{6} T_{ijk}x_j(0)\,x_k(0)$$

where $x_1=x$ $x_2=\theta$ $x_3=y$ $x_4=\varphi$ $x_5=\ell$ and $x_6=\delta$ denotes the subscript notation. In an actual computer printout, the T_{ijk}'s are abbreviated as (i jk); for example $T_{126} = (x|\theta_o\delta)$ would appear in a printout as (1 26) followed by the computed value of the aberration coefficient for the system being designed.

In order to modify the magnitude of any given aberration coefficient, it is necessary to introduce multipole component(s) of the magnetic field of order equal to or less than the order of the aberration. Thus sextupole, quadrupole and dipole components of the field may all be used to modify any given second-order aberration. But, in practice, the second-order aberrations are usually minimized by only introducing sextupole components so as not to disturb the first-order optics of the system. It should always be kept in mind, however, that it may be

beneficial to go back and change the first-order solution (optical mode) so as to provide a more favorable situation for correcting aberrations; a wise selection of the first-order optical mode may in many instances be the deciding factor between the success or failure of a design.

For a fixed location of a sextupole component, the partial derivative of any second-order aberration coefficient T_{ijk} with respect to the strength S_2 of a given sextupole component is a constant. i.e.,

$$\frac{\partial T_{ijk}}{\partial S_2} = \text{a constant} = \text{the coupling coefficient of } S_2 \text{ to } T_{ijk}$$

Thus minimizing a selected group of aberrations is a straight forward problem of solving a set of simultaneous linear equations once the coupling coefficients are known.

The strengths of the sextupole components may be determined directly by TRANSPORT. The user may either constrain certain second order matrix elements to certain values, or may minimize the net second-order contributions to a given component of the beam ellipsoid

Second-Order Phase Ellipsoid Formalism

It will be noted by the user, that a second-order TRANSPORT calculation modifies the phase-ellipsoid printout. In a second-order run, TRANSPORT calculates and prints out the second-moments of the phase space distribution function in the $\sqrt{\sigma_{ii}}$ columns. In addition, it also calculates and prints out the new coordinates of the centroid (first-moment) of the phase space distribution function and tabulates this result to the left of the $\sqrt{\sigma_{ii}}$ columns in the same manner as it does for a magnet misalignment run.

Caution should be used in the use and interpretation of the second-order phase ellipsoid results especially if it is known or suspected that the phase space

distribution resulting from a second-order run is not symmetrical about the beam centroid. To be certain of the situation in any given design, it would be wise to calculate the actual distribution function by using the Monte-Carlo computer program TURTLE[1].

The actual method used in TRANSPORT by which the second-order terms are included in the beam ellipse is described in the following report. The reader should bear in mind that the derivation is based on a gaussian initial beam distribution. For any other initial distribution the second order effects on the beam ellipsoid should be regarded only as an approximation.

[1] D. C. Carey, "TURTLE, A Computer Program for Simulating Charged Particle Beam Transport Systems", N.A.L. Report No. 64, Fermi National Accelerator Laboratory, Batavia, Illinois (1971).

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Second Order Contributions to Beam Dimensions

David C. Carey

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I. Introduction

The phase space region occupied by an aggregate of charged particles in a beam line is often represented by a higher dimensional ellipsoid. Given no further information, one might interpret such an ellipsoid as an envelope inside of which particles are distributed uniformly, or as giving the scale dimensions of a gaussian distribution. The latter case has the advantage that is easily adapted to include higher order effects of the beam line. In either case the parameters of the ellipsoid are simply related to the first and second moments and therefore the width of the distribution in any coordinate. In first order an ellipsoid at any point in a beam line is transformed into another ellipsoid at any other location in a beam line. In second and higher orders a transformation from one location in a beam line to another will cause the ellipsoid to become distorted. One can still, however, calculate the first and second moments of the distribution, and thereby obtain a measure of its dimensions in any coordinate.

Below we elaborate on the methods for calculating the ellipsoid parameters at any point in the beam line. Much of the first order theory can be found in the work of Brown and Howry.[1] It is included here for completeness.

II. The Ellipsoid Formalism

The position and motion of a particle in a beam line may be represented via a six-dimensional vector.

$$x = \begin{pmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{pmatrix} \tag{1}$$

The coordinates x and y represent respectively the horizontal and vertical displacements at the position of the particle, θ and ϕ, the angles with the axis of the beam line in the same planes. The quantity ℓ represents the longitudinal position of the particle relative to a particle traveling on the magnetic axis of the system with the central momentum designed for the system. The remaining quantity $\delta = \frac{\Delta p}{p}$ gives the fractional deviation of the momentum of the particle from the central design momentum of the system.

An ellipsoidal hypersurface in this six-dimensional space may be represented by the equation:

$$x^T \sigma^{-1} x = 1 \tag{2}$$

where σ^{-1} is a symmetric positive definite matrix. We represent this matrix as an inverse for reasons which will become apparent later. At this stage the center of the ellipsoid is assumed to lie at the origin of the coordinate system. The ellipsoid may be taken to be the envelope of a uniform distribution, or the scale in a gaussian distribution, giving a particle density:

$$\rho = C \exp(-\tfrac{1}{2} x^T \sigma^{-1} x) \tag{3}$$

For any real symmetric matrix there exists a coordinate system in which that matrix is diagonal and an orthogonal transformation to that coordinate system.[2] Let us represent the orthogonal transformation by the matrix O, so that:

$$x_i = \sum_j O_{ij} \tilde{x}_j \tag{4}$$

where $\tilde{x}_j$ are the coordinates in the frame where the transform of σ^{-1} and therefore that of σ are diagonal. Calling the matrix σ transformed to the new frame $\tilde{\sigma}$ we now have:

$$\sigma_{ij} = \sum_{k\ell} O_{ik} \tilde{\sigma}_{k\ell} O_{j\ell} \tag{5}$$

and equation (1) becomes

$$\tilde{x}^T \tilde{\sigma}^{-1} \tilde{x} = 1 \tag{1a}$$

Specializing to the gaussian distribution, it is now an easy matter to calculate the second moments in the new frame since the coordinates are decoupled. We arrive at:

$$\overline{\tilde{x}_i \tilde{x}_j} = \tilde{\sigma}_{ij} = \delta_{ij} \tilde{\sigma}_{jj} \tag{6}$$

The second moments in the old frame are now:

$$\overline{x_i x_j} = \overline{\sum_{k\ell} O_{ik} O_{j\ell} \tilde{x}_k \tilde{x}_\ell} = \sum_{k\ell} O_{ik} O_{j\ell} \overline{\tilde{x}_k \tilde{x}_\ell}$$

$$= \sum_{k\ell} O_{ik} O_{j\ell} \tilde{\sigma}_{k\ell} = \sigma_{ij}. \tag{7}$$

Therefore in this case the elements of the matrix σ give the second moments of the distribution in the original coordinate system. The density function, properly normalized, now becomes:

$$\rho = \frac{N_o}{\sqrt{\det(\sigma)\ (2\pi)^3}} \exp(-\tfrac{1}{2} x^T \sigma^{-1} x) \tag{8}$$

where N_o is the total number of particles. Since the matrix O is orthogonal the determinants of σ and $\tilde{\sigma}$ are equal.

The elements of the matrix σ may be put in more convenient form for interpretation. The square roots of the diagonal elements may be taken as giving the half widths x_o of the distribution in a given coordinate while the off-diagonal elements may be related to the correlations r_{ij}, so

$$x_{oi} = \sqrt{\sigma_{ii}}$$

$$r_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}} \tag{9}$$

Since, for any positive definite symmetric matrix σ, we have:[2]

$$\sigma_{ii}\,\sigma_{jj} - \sigma_{ij}^2 > 0 \tag{10}$$

the correlations must all obey the inequality

$$|r_{ij}| < 1 \tag{11}$$

If the ellipsoid is interpreted as describing the envelope of a uniform distribution, then the x_{oi} represent the maximum extents of the beam in the given coordinates.

III. The Effect of a Beam Line

A. First Order

If we now let $x_j^{(1)}$ be the coordinates of a ray at the initial point in a beam line, and $x_i^{(2)}$ the coordinates at some later point, the two are related by the equation:

$$x_i^{(2)} = \sum_j R_{ij}\, x_j^{(1)} \tag{12}$$

If we continue to assume a distribution centered at the origin the first moments at both initial and final point will be zero. The second moments will now be given by:

$$\sigma_{ij}^{(2)} = \overline{x_i^{(2)}\ x_j^{(2)}} = \sum_{k\ell} R_{ik}\ R_{j\ell}\ \overline{x_k^{(1)}\ x_\ell^{(1)}} \tag{13}$$

$$= \sum_{k\ell} R_{ik}\ R_{j\ell}\ \sigma_{k\ell}^{(1)}$$

or more concisely

$$\sigma^{(2)} = R\sigma^{(1)}R^T \tag{14}$$

To first order an ellipsoid at the initial point will transform into an ellipsoid at the final point, so that the equation:

$$x^{(2)T}(\sigma^{(2)})^{-1}\ x^{(2)} = 1 \tag{15}$$

will give the envelope of the particle distribution at the later point.

B. Second Order

In second order the transformation on the coordinates effected by the beam line is given by:

$$x_i^{(2)} = \sum_j R_{ij}\ x_j^{(1)} + \sum_{jk} T_{ijk}\ x_j^{(1)}\ x_k^{(1)}. \tag{16}$$

We employ here a symmetric T matrix whose off-diagonal elements are half those of the T matrix used by Brown. The first and second moments of the distribution at the final point are now given by:

$$\overline{x_i^{(2)}} = \sum_j R_{ij}\ \overline{x_j^{(1)}} + \sum_{jk} T_{ijk}\ \overline{x_j^{(1)}\ x_k^{(1)}} \tag{17}$$

$$\overline{x_i^{(2)}\ x_j^{(2)}} = \sum_{k\ell} R_{ik}\ R_{j\ell}\ \overline{x_k^{(1)}\ x_\ell^{(1)}}$$

$$+ \sum_{k\ell m} \left[R_{ik}\ T_{j\ell m} + T_{ik\ell}\ R_{jm}\right] \overline{x_k^{(1)}\ x_\ell^{(1)}\ x_m^{(1)}}$$

$$+ \sum_{k\ell mn} T_{ik\ell}\ T_{jmn}\ \overline{x_k^{(1)}\ x_\ell^{(1)}\ x_m^{(1)}\ x_n^{(1)}}$$

For a symmetric, on-axis initial distribution, the first and third moments vanish. The problem now reduces to determining the fourth moments of the initial distribution.

As an extension of previous notation we now denote the fourth moments of the distribution about the initial point by $\sigma^{(1)}_{ijk\ell}$. We consider the coordinate system in which the matrix of second moments σ_{ij} is diagonalized, denoting the moments in this frame by $\tilde{\sigma}$. Then from equation (7) we have:

$$\sigma_{ij} = \sum_{k\ell} O_{ik}\, O_{j\ell}\, \tilde{\sigma}_{k\ell} \qquad (18)$$

$$= \sum_{k} O_{ik}\, O_{jk}\, \tilde{\sigma}_{kk}$$

We continue to specialize to a gaussian distribution so that the fourth moments will be directly derviable from the second moments. In the diagonal frame the coordinates separate, and the fourth moments are easily calculated. The only ones which are non-zero are $\tilde{\sigma}_{iijj}$, $\tilde{\sigma}_{ijij}$, or $\tilde{\sigma}_{ijji}$ for $i \neq j$, and $\tilde{\sigma}_{iiii}$ with:

$$\tilde{\sigma}_{iijj} = \tilde{\sigma}_{ii}\, \tilde{\sigma}_{jj} \qquad (19)$$

$$\tilde{\sigma}_{ijij} = \tilde{\sigma}_{ii}\, \tilde{\sigma}_{jj}$$

$$\tilde{\sigma}_{ijji} = \tilde{\sigma}_{ii}\, \tilde{\sigma}_{jj}$$

$$\tilde{\sigma}_{iiii} = 3\, \tilde{\sigma}_{ii}\, \tilde{\sigma}_{ii}$$

so that in general:

$$\tilde{\sigma}_{ijk\ell} = \delta_{ij}\, \delta_{k\ell}\, \tilde{\sigma}_{ii}\, \tilde{\sigma}_{kk} + \delta_{ik}\, \delta_{j\ell}\, \tilde{\sigma}_{ii}\, \tilde{\sigma}_{jj}$$

$$+ \delta_{i\ell}\, \delta_{jk}\, \tilde{\sigma}_{ii}\, \tilde{\sigma}_{jj}. \qquad (20)$$

Now if under the transformation O, the fourth moments transform as:

$$\sigma_{ijk\ell} = \sum_{mnop} O_{im}\, O_{jn}\, O_{ko}\, O_{\ell p}\, \tilde{\sigma}_{mnop} \tag{21}$$

then from equation (1a) we finally arrive at:

$$\sigma_{ijk\ell} = \sigma_{ij}\, \sigma_{k\ell} + \sigma_{ik}\, \sigma_{j\ell} + \sigma_{i\ell}\, \sigma_{jk} \tag{22}$$

Substituting into equation (17) we determine that:

$$\overline{x_i^{(2)}} = \sum_{jk} T_{ijk}\, \sigma_{jk}^{(1)} \tag{23}$$

$$\overline{x_i^{(2)}\, x_j^{(2)}} = \sum_{k\ell} R_{ik}\, R_{j\ell}\, \sigma_{k\ell}^{(1)} + \left(\sum_{k\ell} T_{ik\ell}\, \sigma_{k\ell}^{(1)}\right) \left(\sum_{mn} T_{jmn}\, \sigma_{mn}^{(1)}\right)$$

$$+ 2 \sum_{\ell m} \left(\sum_{k} T_{ik\ell}\, \sigma_{km}^{(1)}\right) \left(\sum_{n} T_{jmn}\, \sigma_{\ell n}^{(1)}\right)$$

Note that, because of the symmetry properties of both T and σ that the two expressions in parentheses in the last term of the second equation represent the same array. From a practical standpoint this means that it needs to be calculated only once.

We see from equation (23) that the centroid of the distribution at the final point no longer coincides with the beam axis. Letting $\sigma^{(2)}$ represent the matrix of second moments about the new centroid we now have:

$$\sigma_{ij}^{(2)} = \overline{x_i^{(2)}\, x_j^{(2)}} - \overline{x_i^{(2)}}\ \overline{x_j^{(2)}} \tag{24}$$

$$= \sum_{k\ell} R_{ik}\, R_{j\ell}\, \sigma_{k\ell}^{(1)}$$

$$+ 2 \sum_{\ell m} \left(\sum_{k} T_{ik\ell}\, \sigma_{km}^{(1)}\right) \left(\sum_{n} T_{jmn}\, \sigma_{\ell n}^{(1)}\right)$$

IV. Off-Axis Initial Distribution

Now consider a gaussian distribution whose center does not

coincide with the beam axis. Letting the coordinates of the centroid by $\overline{x_i^{(1)}}$, we have for the coordinates of a ray:

$$x_i^{(1)} = \overline{x_i^{(1)}} + \xi_i^{(1)} \tag{25}$$

We let the matrix σ represent the moments of the distribution about its centroid so that:

$$\overline{\xi_i^{(1)}\ \xi_j^{(1)}} = \sigma_{ij}^{(1)} \tag{26}$$

$$\overline{\xi_i^{(1)}\ \xi_j^{(1)}\ \xi_k^{(1)}\ \xi_\ell^{(1)}} = \sigma_{ijk\ell}^{(1)}$$

Equation (17) continues to hold for the moments of the distribution about the b eam axis, while equation (22) holds for the moments about the centroid. We must therefore express one set of moments in terms of the other.

Using equations (22), (25), and (26) and applying the first part of equation (24) to the initial distribution, the initial third and fourth moments are given in terms of the initial first and second moments as follows:

$$\begin{aligned}
\overline{x_i^{(1)}\ x_j^{(1)}\ x_k^{(1)}} &= \overline{x_i^{(1)}}\ \overline{x_j^{(1)}\ x_k^{(1)}} + \overline{x_j^{(1)}}\ \overline{x_i^{(1)}\ x_k^{(1)}} \\
&\quad + \overline{x_k^{(1)}}\ \overline{x_i^{(1)}\ x_j^{(1)}} - 2\,\overline{x_i^{(1)}}\ \overline{x_j^{(1)}}\ \overline{x_k^{(1)}} \\
\overline{x_i^{(1)}\ x_j^{(1)}\ x_k^{(1)}\ x_\ell^{(1)}} &= \overline{x_i^{(1)}\ x_j^{(1)}}\ \overline{x_k^{(1)}\ x_\ell^{(1)}} + \overline{x_i^{(1)}\ x_k^{(1)}}\ \overline{x_j^{(1)}\ x_\ell^{(1)}} \\
&\quad + \overline{x_i^{(1)}\ x_\ell^{(1)}}\ \overline{x_j^{(1)}\ x_k^{(1)}} - 2\,\overline{x_i^{(1)}}\ \overline{x_j^{(1)}}\ \overline{x_k^{(1)}}\ \overline{x_\ell^{(1)}}
\end{aligned} \tag{27}$$

Substituting into equation (17) and rearranging terms we arrive at the following expressions for the first and second moments of the distribution at the final point.

$$\overline{x_i^{(2)}} = \sum_j R_{ij}\, \overline{x_j^{(1)}} + \sum_{jk} T_{ijk}\, \overline{x_j^{(1)}\, x_k^{(1)}} \tag{28}$$

$$\overline{x_i^{(2)}\, x_j^{(2)}} = \sum_{k\ell} R_{ik}\, R_{j\ell}\, \overline{x_k^{(1)}\, x_\ell^{(1)}} + \overline{x_i^{(2)}}\, \overline{x_j^{(2)}} - 2\, X_i^{(2)}\, X_j^{(2)}$$

$$+ 2 \sum_{\ell m} \left(R_{i\ell}\, \overline{x_m^{(1)}} + \sum_k T_{ik\ell}\, \overline{x_k^{(1)}\, x_m^{(1)}}\right) \left(R_{jm}\, \overline{x_\ell^{(1)}} + \right.$$

$$\left. \sum_n T_{jmn}\, \overline{x_\ell^{(1)}\, x_n^{(1)}}\right) - \left(\sum_k R_{ik}\, \overline{x_k^{(1)}}\right) \left(\sum_m R_{jm}\, \overline{x_m^{(1)}}\right)$$

where

$$X_i^{(2)} = \sum_k R_{ik}\, \overline{x_k^{(1)}} + \sum_{k\ell} T_{ik\ell}\, \overline{x_k^{(1)}}\, \overline{x_\ell^{(1)}}$$

is the image of the original centroid.

We may now again use equations (9) and (24) to relate this matrix of second moments to the final beam half widths and correlations.

References

1. Karl L. Brown, Sam K. Howry, SLAC Report No. 91 (1970).

2. F. R. Gantmacher, The Theory of Matrices, Chelsea Publishing Co., New York (1959).

A Systematic Procedure for Designing High Resolving Power Beam Transport Systems or Charged Particle Spectrometers

The following is a report submitted to the Third International Magnet Symposium held in Hamburg, Germany - May 1970. It is a general description of a suggested procedure for designing systems to any order and includes the derivation of the coupling coefficient of an nth-order multipole to any nth-order aberration coefficient. The report also derives the multipole strengths for the three techniques for introducing multipole components into a system: namely, 1) pure multipole fields, 2) non-uniform fields, and 3) contoured entrance or exit boundaries of magnets.

The notation used in this report is identical to TRANSPORT notation except for the following:

Replace x' and y' in the report by θ and φ respectively to convert to TRANSPORT notation.

REPORT SUBMITTED TO THE THIRD INTERNATIONAL MAGNET SYMPOSIUM
HELD IN HAMBURG, GERMANY-May 1970

by

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Summary

By extrapolating the systematics of the general first- and second-order theory of beam transport optics (1,2,3) to include higher order multipole terms, it has been possible to evolve a simple, step by step, procedure for the design of high resolving power static-magnetic beam transport systems. The choice of the appropriate dipole and quadrupole elements for a given system may be determined once the resolving power, solid angle, momentum range and detector system of the instrument have been specified. The partial derivative of any nth-order aberration coefficient with respect to an nth-order multipole component located anywhere in the system has been derived. From this "coupling coefficient", the strength and the optimum location of multipole element(s) to correct or modify a given aberration or group of aberrations is uniquely determined.

I. Introduction

Within the last two decades, significant advances have been made in the understanding of charged particle optics. Perhaps the first major contribution was the development of the theory of the Alternating Gradient Synchrotron (A.G.S.) by Courant, Livingston, and Snyder(4) which led to the first-order matrix algebra formulation of beam-transport optics. Subsequent to this a second-order matrix algebra was developed by Brown, Belbeoch, and Bounin(5); followed by the development at SLAC of the digital computer program called TRANSPORT(6) that is widely used today in many laboratories for solving first- and second-order static-magnetic beam transport problems. In principle, the second-order matrix formalism may be extended to any order, but in practice this approach has proved to be too cumbersome. Thus beyond second-order it has been more efficient to use computer ray-tracing programs which integrate the basic differential equation of motion of the charged particles through the known or assumed magnetic fields. The fundamental difficulty with ray-tracing has been the required computational time to complete a design involving the minimization of many higher-order aberrations.

In this report, we will outline a systematic procedure for the design of high-resolution systems based upon the extrapolation of the first- and second-order theory (1,2,3) to include higher-order multipole components. A general equation has been derived for the coupling coefficient of an nth-order multipole to any given nth-order aberration coefficient. As will be shown later, these coupling coefficients are a function only of the characteristic first-order trajectories (matrix elements) introduced and defined in References 1 and 2.

Given this information, a systematic procedure for designing high resolution beam transport systems is as follows:

1) Find a satisfactory first-order solution to the problem using TRANSPORT or its equivalent.

2) Calculate and make the necessary corrections to the second-order aberrations by introducing sextupole components into the system. The "best" locations and strengths of the sextupole components required may be selected via the coupling coefficients for the aberrations to be minimized.

3) Calculate and make the necessary corrections (via ray-tracing) to the third-order aberrations by introducing octupole components into the system. (Note that an nth-order multipole couples with terms of order n or higher but not with terms of order lower than n. Thus an octupole component will not disturb the first- and second-order solutions already found from steps 1 and 2.)

4) Repeat the above procedure up to the multipole order desired or needed to achieve the design objectives.

If the design requires a solution to nth-order and m multipoles at each order are necessary to minimize the aberrations, the number of computer runs previously needed to complete a design was at least $(n+m)^2$. Having a knowledge of the coupling coefficients, after the first-order design has been selected, now (in principle) reduces the number of computer runs required to n. Since ray-tracing is very time consuming, this is indeed a significant saving.

II. Theory*

The following results are applicable to static-magnetic charged particle optical systems possessing median plane symmetry. As in Ref. 1, we shall use a right-handed curvilinear coordinate system (x,y,t) where x and y are the transverse coordinates. x is the outward normal distance in the median plane away from the central trajectory, y is the perpendicular distance from the median plane, t is the distance along the central trajectory, and h=h(t) is the curvature of the central trajectory.

The existence of the median plane requires that the scalar potential φ be an odd function of y, i.e., $\varphi(x,y,t) = -\varphi(x,-y,t)$. The most general form of φ may therefore, be expressed as follows:

$$\varphi(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{2m+1,n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \tag{1}$$

where the coefficients $A_{2m+1,n}$ are functions of t.

In this coordinate system, the differential line element dT is given by

$$dT^2 = dx^2 + dy^2 + (1+hx)^2\, dt^2$$

The Laplace equation has the form

$$\nabla^2\varphi = \frac{1}{(1+hx)} \frac{\partial}{\partial x} \left[(1+hx) \frac{\partial\varphi}{\partial x}\right] + \frac{\partial^2\varphi}{\partial y^2} + \frac{1}{(1+hx)} \frac{\partial}{\partial t} \left[\frac{1}{(1+hx)} \frac{\partial\varphi}{\partial t}\right] = 0 \tag{2}$$

Substitution of (1) into (2) gives the following recursion formula for the coefficients:

$$\begin{aligned} -A_{2m+3,n} = {} & A''_{2m+1,n} + nhA''_{2m+1,n-1} - nh'A'_{2m+1,n-1} + A_{2m+1,n+2} \\ & + (3n+1)hA_{2m+1,n+1} + n(3n-1)h^2A_{2m+1,n} + n(n-1)^2h^3A_{2m+1,n-1} \\ & + 3nhA_{2m+3,n-1} + 3n(n-1)h^2A_{2m+3,n-2} + n(n-1)(n-2)h^3A_{2m+3,n-3} \end{aligned} \tag{3}$$

where prime means $\frac{d}{dt}$, and where it is understood that all coefficients A with one or more negative subscripts are zero. This recursion formula expresses all

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* The notation used in this report follows that used in Ref. 1 unless otherwise indicated.

the coefficients in terms of the midplane field $B_y(x,o,t)$:

where

$$A_{1,n} = \left(\frac{\partial^n B_y}{\partial x^n}\right)_{\substack{x=o \\ y=o}} = \text{functions of } t. \tag{4}$$

Since φ is an odd function of y, on the median plane we have $B_x = B_t = 0$. The normal (in x direction) derivatives of B_y on the reference curve defines B_y over the entire median plane, hence the magnetic field $\vec{B}$ over the whole space. The components of the field are expressed in terms of φ explicitly by $\vec{B} = \vec{\nabla}\varphi$ or

$$B_x = \frac{\partial\varphi}{\partial x} = \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} A_{2m+1,n+1}\ \frac{x^n}{n!}\ \frac{y^{2m+1}}{(2m+1)!}$$

$$B_y = \frac{\partial\varphi}{\partial y} = \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} A_{2m+1,n}\ \frac{x^n}{n!}\ \frac{y^{2m}}{(2m)!}$$

$$B_t = \frac{1}{(1+hx)}\ \frac{\partial\varphi}{\partial t} = \frac{1}{(1+hx)}\ \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} A'_{2m+1,n}\ \frac{x^n}{n!}\ \frac{y^{2m+1}}{(2m+1)!} \tag{5}$$

The expression for the magnetic field on the midplane is

$$B_y(x,o,t) = \sum_{n=o}^{\infty} A_{1,n}\ \frac{x^n}{n!} \tag{6}$$

At this point we deviate from the notation and formalism of Ref. 1 and introduce $K_n(t)$, the multipole strength per unit length; and S_n, the total multipole strength of a static-magnetic field.

We rewrite equation (6) as

$$B_y(x,o,t) = B\rho \sum_{n=o}^{\infty} K_n(t)\ x^n \tag{7}$$

Where $B\rho = \frac{B}{h} = \frac{P_o}{e}$ is the magnetic rigidity of a particle of momentum P_o and charge e along the central trajectory; from which

$$K_n(t) = \left(\frac{1}{B\rho}\right)\left(\frac{1}{n!}\right)\left(A_{1,n}(t)\right) = \left(\frac{1}{B\rho}\right)\left(\frac{1}{n!}\right)\left(\frac{\partial^n B_y}{\partial x^n}\right)_{x=y=o} \tag{8}$$

We define S_n as

$$S_n = \int_0^L K_n(t)dt \tag{9}$$

S_n so defined is the strength of the nth-order multipole component of a field over the interval of integration.

Multipole Strengths for Pure Multipole Fields

Consider the scalar potential of an nth-order $\left[2(n+1)\text{pole}\right]$ pure multipole element:

$$\varphi = \frac{B_o r^{n+1}}{(n+1)a^n} \left[\sin (n+1)\ \theta\right] \tag{10}$$

where

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

B_o is the field at the pole and a is the radial distance to the pole from the central trajectory.

Expanding φ as a function of x and y and differentiating, we have

$$B_y = \frac{\partial\varphi}{\partial y} = \frac{B_o}{a^n}\left[x^n + \cdots\cdots\right]$$

From which

$$K_n = \left(\frac{B_o}{a^n}\right)\left(\frac{1}{B\rho}\right)$$

and

$$S_n = \left(\frac{B_o}{a^n}\right)\left(\frac{L}{B\rho}\right) \tag{11}$$

Where L is the length of the multipole element.

For a dipole n=o and the dipole strength is

$$S_o = \frac{L}{\rho} = \alpha \text{ (The angle of bend of the central trajectory)}$$

For a quadrupole n=1 and

$$S_1 = \left(\frac{B_o}{a}\right)\left(\frac{L}{B\rho}\right)$$

For a sextupole n=2 and

$$S_2 = \left(\frac{B_o}{a^2}\right)\left(\frac{L}{B\rho}\right)$$

etc. for higher-order multipoles.

Multipole Strengths for a Non-Uniform Field Expansion

From the midplane field expansion of a non-uniform magnetic field

$$B_y(x,o,t) = B_y(o,o,t)\left[1-nhx+\beta(hx)^2+\gamma(hx)^3+ \cdots\cdots\right] \qquad (12)$$

the multipole strength factors are:

$$K_o = h, \quad K_1 = -nh^2, \quad K_2 = \beta h^3, \quad \text{etc.}$$

and S_n evaluated over the length L of the central trajectory is:

$$S_o = hL = \alpha \text{ as before,}$$

$$S_1 = -nh^2L, \quad \text{and} \quad S_2 = \beta h^3 L, \quad \text{etc.}$$

Multipole Strengths for a Contoured Entrance or Exit Boundary of a Magnet

A third method of introducing multipole components is via a curved entrance or exit boundary of a magnet. To calculate the multipole strengths in this case, we integrate equation (7), holding x constant, as follows:

$$\int_o^L B_y(x,o,t)dt = B\rho \sum x^n \int_o^L K_n(t)dt = B\rho \sum S_n x^n \qquad (13)$$

To relate this to the field boundary, we assume B_y to be a constant inside the effective field boundary and zero outside (i.e., we ignore the finite extent of the fringing field). In this sharp-cutoff approximation, the field boundary $Z = Z(x)$ is:

$$Z = \frac{1}{B_y} \int_0^L B_y(x,o,t)dt = \frac{1}{h} \sum S_n x^n = - x \tan \beta + \frac{S_2}{h} x^2 + \cdots \qquad (14)$$

where $h = \frac{1}{\rho}$ and β is the angle of rotation of the entrance or exit face of the magnet at x=o. A positive β implies radial (x) defocusing and transverse (y) focusing. We note that:

$$S_1 = - h \tan \beta = \text{The "quadrupole strength"}$$

The radius of curvature of the boundary is related to the sextupole strength as follows:

$$\frac{1}{R} = \frac{Z''}{(1+Z'^2)^{3/2}} = \frac{2 S_2}{h \sec^3 \beta}$$

or

$$S_2 = \frac{h \sec^3 \beta}{2R} = \text{The "sextupole strength"}$$

From equation (13), we note that a positive multipole component of the field increases the $\int Bdt$ for a positive x; thus a positive sextupole is represented by a concave surface of the entrance or exit boundary.

The Description of the Trajectories as a Taylor's Expansion

The deviation of an arbitrary trajectory from the central trajectory is described by expressing x and y as functions of t. The expressions will also contain x_o, y_o, x'_o, y'_o and δ, where the subscript o indicates that the quantity is evaluated at t=o. The prime (') denotes the derivative with respect to t, and $\delta = \frac{\Delta P}{P_o}$ is the fractional momentum deviation of the ray from that of the central trajectory. These five initial boundary values will have the value zero for the central trajectory itself. x and y are expressed as a five-fold Taylor expansion using these initial boundary values. The expansions are written:

$$x(t) = \sum (x|x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\upsilon}\, \delta^{\chi}) x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\nu}\, \delta^{\chi}$$

$$y(t) = \sum (y|x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\upsilon}\, \delta^{\chi}) x_o^{\kappa}\, y_o^{\lambda}\, x_o'^{\mu}\, y_o'^{\nu}\, \delta^{\chi} \qquad (15)$$

Here, the parentheses are symbols for the Taylor coefficients; the first part of the symbol identifies the coordinate represented by the expansion, and the second indicates the term in question. These coefficients are functions of t to be determined. The symbol $\sum$ indicates summation over zero and all positive integer values of the exponents κ, λ, μ, ν, χ; . The constant term is zero, and the terms that would indicate a coupling between the coordinates x and y are also zero; this results from the midplane symmetry. Thus we have

$$\begin{aligned} (x|1) &= (y|1) = 0 \\ (x|y_o) &= (y|x_o) = 0 \\ (x|y_o') &= (y|x_o') = 0 \end{aligned} \qquad (16)$$

Here, the first line is a consequence of choosing the central trajectory as the reference axis, while the second and third lines follow directly from considerations of median plane symmetry.

Since they will appear often in the formalism, it is convenient to introduce the following abbreviations for the first-order Taylor coefficients:

$$(x|x_o) = c_x(t) \qquad (x|x_o') = s_x(t) \qquad (x|\delta) = d_x(t)$$

$$(y|y_o) = c_y(t) \qquad (y|y_o') = s_y(t) \qquad (17)$$

When the transverse position of an arbitrary trajectory at position t is written as a first-order Taylor's expansion as a function of the initial boundary conditions, the above five quantities are just the coefficients appearing in the expansion for the transverse coordinates x and y as follows:

$$x(t) = c_x(t)\, x_o + s_x(t)\, x_o' + d_x(t)\, \delta + \text{higher-order terms}$$

and

$$y(t) = c_y(t)\, y_o + s_y(t)\, y_o' + \text{higher-order terms}.$$

III. Solution of the Equations of Motion

The general differential equation of motion of a charged particle in a static-magnetic field valid to all orders in x and y and their derivatives as derived in Ref. 1, equation (5) is:

$$\hat{x}\left\{[x'' - h(1+hx)] - \frac{x'}{(T')^2}\,[x'x'' + y'y'' + (1+hx)(hx'+h'x)]\right\}$$

$$+ \hat{y}\left\{y'' - \frac{y'}{(T')^2}\,[x'x'' + y'y'' + (1+hx)(hx'+h'x)]\right\}$$

$$+ \hat{t}\left\{(2hx'+h'x) - \frac{(1+hx)}{(T')^2}\,[x'x'' + y'y'' + (1+hx)(hx'+h'x)]\right\}$$

$$= \frac{e}{P}\,T'(\vec{T}' \times \vec{B}) = \frac{e}{P}\,T'\left\{\hat{x}[y'B_t - (1+hx)B_y] + \hat{y}[(1+hx)B_x - x'B_t]\right.$$

$$\left. + \hat{t}[x'B_y - y'B_x]\right\} \qquad (18)$$

If this equation is solved to nth-order for the Taylor's coefficients of equation (15), it will be observed that the result has the remarkably simple form:

$$(x_i|x_o^{\kappa}y_o^{\lambda}x_o'^{\mu}y_o'^{\nu}\delta^{\chi}) = \pm\left[\frac{n!}{\kappa!\lambda!\mu!\nu!\chi!}\right]\int_o^t G_i(t,\tau)c_x^{\kappa}(\tau)c_y^{\lambda}(\tau)s_x^{\mu}(\tau)s_y^{\nu}(\tau)d_x^{\chi}(\tau)K_n(\tau)d\tau$$

$$+ \text{Terms containing } K_o, \cdots\cdots\cdots K_{n-1} \qquad (19)$$

where the variable of integration is τ and $n = (\kappa + \lambda + \mu + \nu + \chi)$.

The x_i have the following meaning:

$$x_1 = x(t) \qquad x_2 = x'(t) \qquad x_3 = y(t) \qquad x_4 = y'(t)$$

c_x, c_y, s_x, s_y, and d_x are defined by equation (17) and in general are functions of the variable of integration τ over the interval of integration. K_n is defined by equation (8) and in general is also a function of τ.

The G_i's are Green's functions where:

$$G_1(t,\tau) = \left(x(t)|x'(\tau)\right) = s_x(t)c_x(\tau) - c_x(t)s_x(\tau)$$

$$G_2(t,\tau) = \left(x'(t)|x'(\tau)\right) = s_x'(t)c_x(\tau) - c_x'(t)s_x(\tau)$$

$$G_3(t,\tau) = \left(y(t)|y'(\tau)\right) = s_y(t)c_y(\tau) - c_y(t)s_y(\tau)$$

$$G_4(t,\tau) = \left(y'(t)|y'(\tau)\right) = s_y'(t)c_y(\tau) - c_y'(t)s_y(\tau) \qquad (20)$$

Note that the G_i's are just first-order Taylor's coefficients measured from the location (τ) of the multipole component to the end of the system (t).

Thus we see that the coupling coefficient to an nth-order multipole is a function only of the first-order matrix elements c_x, c_y, s_x, s_y, d_x and their derivatives with respect to t.

From median-plane symmetry considerations, the allowed aberrations are those with y and/or y' appearing an even number of times in the Taylor coefficient. For example $(x|x_o^2)$, $(x|y_o y_o')$ and $(y|y_o^2 y_o')$ are allowed aberrations; whereas $(x|y_o)$, $(x|x_o^2 y_o')$ or $(y|y_o^2)$ are not allowed and are therefore equal to zero.

The minus sign is used when y and/or y' appear 0, 4, 8, 12 $\cdots\cdot$ times and the plus sign is used when y and/or y' appear 2, 6, 10 $\cdots\cdots$ times. For example for the coefficients $(x|x_o^2)$ and $(y|y_o^3)$, the minus sign is applicable; whereas for the coefficients $(x|y_o^2)$ and $(y'|y_o^3 y_o'^2)$ the plus sign is applicable.

Equation (19) is derived by observing in the pattern of the solution of the differential equation that an nth-order aberration term containing the nth-order multipole strength factor K_n cannot include multipole strength factors of lower order than n; or stated physically, an nth-order multipole cannot couple to aberrations (terms) of order lower than n. This fact allows the recursion formula equation (3) to be reduced to the simple form

$$A_{2m+3,n} = -A_{2m+1,n+2} \qquad (21)$$

in so far as it applies to the derivation of nth-order terms containing only K_n. As a consequence, the scalar potential for deriving these terms assumes the simplied form

$$\Phi(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m A_{1,2m+n} \frac{x^n}{n!} \frac{y^{2m+1}}{(2m+1)!} \qquad (22)$$

From which, it follows that

$$B_x(x,y,t) = \left(\frac{P_o}{e}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^m K_{2m+n+1} \frac{(2m+n+1)!}{n!(2m+1)!} x^n y^{2m+1}$$

and

$$B_y(x,y,t) = \left(\frac{P_o}{e}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}' (-1)^m K_{2m+n} \frac{(2m+n)!}{n!\ 2m!} x^n y^{2m} \qquad (23)$$

For terms containing only K_n, the basic differential equations assume the form:

$$x'' + \cdots\cdots = -\frac{e}{P} B_y$$

$$y'' + \cdots\cdots = \frac{e}{P} B_x \qquad (24)$$

Substituting the Taylor's expansion of equation (15) and solving for the nth-order terms using a conventional Green's function solution (see Ref. 1) yields equation (19) above.

IV. Interpretation and Use of Equation (19)

For most practical cases of interest, K_n will be a constant over the interval of integration. In this event we may define the coupling coefficient of an nth-order multipole to an nth-order aberration as the partial derivative of equation (19) with respect to the K_n in question as follows:

$$\frac{\partial(x_i|x_o^{\kappa} y_o^{\lambda} x_o'^{\mu} y_o'^{\nu} \delta^{\chi})}{\partial K_n} = \pm \left[\frac{n!}{\kappa!\lambda!\mu!\nu!\chi!}\right] \int_0^L G_i c_x^{\kappa} c_y^{\lambda} s_x^{\mu} s_y^{\nu} d_x^{\chi} d\tau \qquad (25)$$

where now the interval of integration is over the multipole length L represented by K_n. For a distributed multipole component (such as in a non-uniform field bending magnet), equation (25) is used.

In many cases where a curved entrance or exit pole contour is used or a short multipole magnet is used such that the characteristic first-order functions c_x, c_y, s_x, s_y and d_x are essentially constants over the interval of integration (the length of one multipole), then the coupling coefficient is best defined as the partial derivative of equation (19) with respect to S_n as follows:

$$\frac{\partial(x_i | x_o^{\kappa} y_o^{\lambda} x_o'^{\mu} y_o'^{\nu} \delta^{\chi})}{\partial S_n} = \pm \left[\frac{n!}{\kappa!\lambda!\mu!\nu!\chi!}\right] G_i c_x^{\kappa} c_y^{\lambda} s_x^{\mu} s_y^{\nu} d_x^{\chi} \tag{26}$$

Examples

Assume a situation where the end of the system is a point-to-point image or the origin (i.e., $s_x(t)=o$), then using equation 26, the coupling coefficients of a sextupole of strength S_2 to various second order aberration coefficients are:

$$\frac{\partial(x| x_o'\delta)}{\partial S_2} = c_x(t)\, s_x^2 d_x$$

$$\frac{\partial(x| x_o'^2)}{\partial S_2} = c_x(t)\, s_x^3$$

$$\frac{\partial(x| y_o y_o')}{\partial S_2} = -2\, c_x(t)\, c_y\, s_x\, s_y \tag{27}$$

etc. Where the Green's function used in these examples is

$$G_1 = s_x(t)c_x - c_x(t)s_x = -c_x(t)s_x \quad \text{(since } s_x(t) = 0 \text{ for point-to-point imaging)}$$

The aberration and $c_x(t)$ are evaluated at the end of the system. $c_x(t)$ is equal to the magnification M_x in the examples given. The remaining coefficients c_y, s_x, s_y and d_x are evaluated at the location of the sextupole S_2. The above results are in agreement with Table VII of Ref. 1.

To illustrate a more complex example, consider the fourth-order aberration coefficient $(y|y_o^2 y_o'\delta)$ and assume parallel-to-point imaging in the y coordinate (i.e., $c_y(t) = 0$). The appropriate Green's function is:

$$G_3 = s_y(t)c_y - c_y(t)s_y = s_y(t)c_y$$

and the coupling coefficient to a fourth-order multipole of strength S_4 is:

$$\frac{\partial(y| y_o^2 y_o'\delta)}{\partial S_4} = -\left(\frac{4!}{2!}\right) s_y(t) c_y^3 s_y d_x \tag{28}$$

where again the aberration coefficient $(y|y_o^2 y'_o \delta)$ and $s_y(t)$ are evaluated at the end of the optical system and c_y, s_y, and d_x are evaluated at the location of the fourth-order multipole S_4.

V. A Systematic Procedure for Designing High Resolution Systems

First-Order Considerations

In many respects, the determination of a satisfactory first-order magnetic-optical design is more difficult to achieve than is the subsequent higher-order design. This is true not only because the basic equipment configuration is dominated by first-order optical considerations but also because the choice of the first-order optics affects the magnitude of all higher-order aberrations and the ease with which these aberrations may be minimized by introducing multipole components into the design.

The dominating design parameters that must be clearly specified in order to evolve a first-order design are the momentum resolving power; the spatial resolution of the particle detector system to be used (this determines the momentum dispersion required); the required phase space acceptance (the solid angle, the source size, and the momentum range) of the instrument, and the first-order imaging requirements in both the x and y coordinates.

Given the above specifications (assuming they are self-consistent), the optical mode and physical configuration of the instrument may be determined. Often, more than one theoretical solution exists; in which case the choice is usually resolved by practical or economic considerations. In other cases, no solution is evident and the basic specifications must be modified accordingly. In any event, the following equations and discussion are applicable to the solution of the problem.

1) First-Order Resolving Power

A general equation for the first-order resolving power has been derived in References (1,2, and 3). For point-to-point imaging the first-order momentum resolving power R_1 is defined as the ratio of the momentum dispersion at the image plane to the total image size. Thus if $2x_o$ is the total source size then from Reference 1 we have:

$$R_1 = \frac{P}{\Delta P} = \left| \frac{d_x(t)}{2x_o c_x(t)} \right| = \frac{1}{2x_o} \left| \int_o^t s_x(\tau) h(\tau) d\tau \right| \qquad (29)$$

Note that $h(\tau)d\tau = d\alpha$ is the differential angle of bend of the central trajectory of the optical system.

Equation (29) may be expressed in a number of useful forms. If we consider a particle originating at the source with $x_o = o$ and $\delta = \frac{\Delta P}{P_o} = o$ and lying in the midplane (i.e., a monoenergetic point source), the first-order equation of its trajectory is

$$x(\tau) = s_x(\tau)x'_o \tag{30}$$

We may then rewrite equation (29) as follows:

$$R_1 = \frac{1}{2x_o x'_o}\left|\int_o^t x(\tau)h(\tau)d\tau\right| = \frac{(\ell-\ell_o)}{2x_o x'_o} \tag{31}$$

where $(\ell-\ell_o)$ is the path length difference between the trajectory described by equation (30) and the central trajectory. Or we may also write equation (31) in the form

$$R_1 = \frac{1}{2x_o x'_o}\left|\int_o^t \frac{B\,x(\tau)d\tau}{B\rho}\right| = \left(\frac{1}{2x_o x'_o}\right)\left(\frac{1}{B\rho}\right)\left|\int_o^t BdA\right| \tag{32}$$

where $\int BdA$ is the magnetic flux enclosed between the central trajectory and the trajectory described by equation (30), and $B\rho$ is the magnetic rigidity of the central trajectory. Please note, however, that if the trajectory of equation (30) crosses the central trajectory or the sign of B changes, this changes the sign of the integration. From equation (32) we may define resolving power as the magnetic flux enclosed per unit phase space area $(2x_o x'_o)$, per unit momentum $(B\rho)$ of the central ray.

In any given design, one or more of the above equations may be used as a guide toward achieving the required resolving power. One of the design decisions that must be made is the appropriate choice of the dipole magnet parameters (width and length) to achieve the required $\int BdA$. From first-order considerations, this choice is dominated primarily by practical and economic factors. However, a study of the nature of the origin of aberrations (see for example Ref. 1) suggests that it is advisable to keep the amplitude of s_x small. In order to simultaneously satisfy this requirement and meet the required resolving power R_1, we see from equation (29) that the total angle of bend α of the central trajectory should be chosen as large as is practical. Also, in general, the focal plane angle tends to be more normal to the optic axis for larger α - a property usually desired in most designs.

2) Dispersion

From Reference 1, 2, or 3; for point-to-point imaging ($s_x(t)=o$) the dispersion at the image plane is

$$d_x(t) = - c_x(t) \int_o^t s_x(\tau)h(\tau)d\tau \qquad (33)$$

where $c_x(t)$ is the magnification at the image plane.

The dispersion and hence the magnification in the design of a spectrometer is dominated almost entirely by a compromise between the spatial resolution of the particle detectors used at the image plane and the momentum range to be covered by the instrument; or in the case of a momentum defining (analyzing) system, by the acceptable momentum-defining slit spacings.

3) The Selection of the Optical Mode

By optical mode, we mean the type of imaging (e.g., point-to-point or parallel-to-point, etc.) required at the image plane in both the x and y coordinates, and the number of intermediate images imposed between the source and image planes. The imaging requirements at the image plane are usually dominated by the physics to be performed by the instrument and the nature of the particle detectors used. However often (especially at low energies) the imaging in the y plane may be unimportant as far as the physics requirements are concerned which in turn provides some additional flexibility in the optics design.

A study of the coupling coefficients to the aberration coefficients (equation 19) shows the not surprising result, that multipoles located at intermediate images in a system do not couple to aberrations in the plane in which the intermediate image occurs. Hence it often proves beneficial to intentionally create an intermediate image in the y plane of an optical system so as to achieve some degree of "orthogonality" in the minimizing of x and y aberrations.

The considerations of 1), 2), and 3) above are the determining factors in the selection of the first-order solution of a system design.

The optical mode and dispersion of the system are determined to a great extent by the choice of the quadrupole components chosen to achieve the first-order imaging although it is clear that the dipole elements also influence the first-order imaging to a greater or lesser extent depending upon the total angle of bend of the system.

4) Aberrations and their Correction

A study of the source of second- and higher-order aberrations (see for example Ref. 1) suggests that it is advisable to maintain the characteristic

first-order functions c_x, s_x, d_x and c_y, s_y and their derivatives as small as is feasible through the magnetic elements of a system when choosing the first-order design. This procedure will tend to reduce the initial size of the aberrations and hence simplify the problem of minimizing them by the addition of multipole components to the system design.

The procedure for minimizing aberrations has already been outlined in the Introduction and as such will not be repeated here. The "key" to the minimization procedure is the coupling coefficient given by the integral expression in equation (19). The "best" location for the correcting multipole is where the coupling coefficient has its maximum value.

The preferred method of introducing the multipole components, i.e., via pure multipoles, contoured entrance or exit boundaries, or non-uniform fields is a combination of practical and economical considerations and, of course, personal taste and experience. All three methods have been used with pure multipoles dominating the situation for higher energy physics and the other two methods dominating medium and low-energy physics applications. All three techniques should be considered in any given design situation to be certain that an important economic or practical advantage has not been ignored.

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