THE EFFECT OF BEAM LINE MAGNET MISALIGNMENTS

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The misalignment of a magnet in a beam line will cause an alteration of the beam envelope at any later point in that beam line. The position of a misaligned magnet may be described in terms of six coordinates, three translational and three rotational. The effect of a misalignment on a single particle trajectory is derived to first order, including bilinear terms. A bilinear effect is one which affects the beam line focusing characteristics, but not the central ray, such as the effect of rotating a quadrupole about its axis. The effect on the beam envelope is calculated, both for a known magnet displacement and for an uncertain magnet position. The formalism has been included in the computer program TRANSPORT.[1]

*Operated by Universities Research Association Inc. under contract with the Energy Research and Development Administration

I. Introduction

The effects of magnet misalignments are an important consideration at every stage of beam line design, installation, and operation. The selection of the optical mode, determination of surveying accuracy requirements, and the choice of correcting elements are all dependent on misalignment information.

Two types of misalignment information are typically needed. To assess the general effect of misalignments in the design stage, one needs to know the change in beam position and beam line transmission characteristics due to uncertainties in the position of each magnetic element in each seperate coordinate. Secondly, to provide for correcting elements, one needs to know the effect on the beam of specific misalignments.

In the following we derive a method of determining the effect of magnet misalignments on a particle beam. We first define a reference system in which to express misalignments. Then we determine the effect of a misalignment on individual particle trajectories. Finally we express the effect on the beam envelope which describes the ensemble of particles comprising the beam.

II. Particle Trajectory Coordinates

To specify the position and direction of a particle at any instant in time, we employ a coordinate system defined with respect to the beam line reference trajectory.[2] The z axis is taken to point along the reference trajectory; the x axis points to the left, and the y axis points up. The position and direction of the particle trajectory can then be given by a vector with six components

$$X = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{pmatrix} \tag{1}$$

The quantities x, x', y, and y' are respectively the horizontal displacement and slope, and vertical displacement and slope of the ray with respect to the central reference trajectory. ℓ is the longitudinal seperation of the ray from a ray which enters the beam line at the same time as the given ray and travels along the central trajectory. δ is the fractional momentum deviation of the particle from the design momentum of the beam line.

When a charged particle passes through a perfectly aligned magnet, the transformation may be described to first order by the matrix equation[2]

$$X(1) = R\,X(0) \tag{2}$$

The sets of six coordinates X(0) and X(1) give the particle position and direction at the entrance and exit faces of the magnet respectively.

When a magnet is misaligned, the central trajectory of the magnet is no longer continuous with the central trajectory of the beam line (see figure 1 below). In particular, at both the entrance and exit faces, the reference coordinate system external to the magnet no longer coincides with the reference coordinate system internal to the magnet (see figure 2 below). The misaligned and aligned reference coordinate systems are related by a translation of origin plus a rotation of axes.

We continue to use X(0) and X(1) to denote respectively the entrance and exit face ray coordinates in the aligned coordinate systems. We use a subscript f to denote the ray coordinates $X_f(0)$ and $X_f(1)$ expressed in the misaligned reference coordinate systems. To first order the ray coordinates in the misaligned coordinate systems may be expressed in terms of those in the aligned coordinate systems by an affine transformation

$$X_f(0) = S_o\, X(0) - D_o \qquad (3)$$

$$X_f(1) = S_1\, X(1) - D_1 \qquad (4)$$

The symbols S_o and S_1 represent six by six matrices, whose form will be derived below. The two six-vectors D_o and D_1 are translations in the six dimensional space of particle coordinates. The three vectors $\overline{D}_o$ and $\overline{D}_1$ formed from the displacement coordinates (x, y, and z) of D_o and D_1 give the displacement due to the misalignment of the origins of the reference coordinate systems. These two three-vectors are shown in figure 1.

III. Magnet Misalignment Coordinates

The alignment of a rigid magnet has six degrees of freedom, three translational and three rotational. These are conveniently represented by the six quantities

$$m = \begin{pmatrix} \delta x \\ \theta_x \\ \delta y \\ \theta_y \\ \delta z \\ \theta_z \end{pmatrix} \tag{5}$$

where δx, δy, δz are the displacements in the x, y, and z directions, and θ_x, θ_y, and θ_z are the rotations about the x, y, and z axes respectively. The origin of the xyz coordinate system, called the pivot, is the point about which the misalignments are measured. If the pivot point is located at some point on the reference trajectory, the x, y, and z axes of the alignment coordinate system are taken to coincide with the x, y, and z axes of the beam line reference coordinate system.

The misalignments form a mathematical group, which is the Euclidean group in three dimensions. This group is non-commutative and the order in which the misalignments are imposed is important if terms of higher order than linear are included. In practice, however, misalignment values are sufficiently small so that a first-order approach is justified. For these reasons, we consider only those terms which are of first order in the misalignment parameters.

IV. Transformation of Particle Trajectory Coordinates

We now temporarily delete the indices 0 and 1 indicating the entrance and exit magnet faces respectively, and consider the effect of a misalignment at a single magnet face. Later we will combine the results from the two faces to obtain the net effect of a misalignment.

When the components of the misalignment vector m are small, we may expand the matrix S and the centroid displacement D in the misalignment parameters. Retaining only first-order terms we have

$$D = Am$$
$$S = I + Bm \qquad (6)$$

The six by six matrix A represents a transformation from the misalignment parameters to the particle coordinates. I is the identity matrix, and B represents a set of six matrices, one for each of the misalignment parameters. A single six by six matrix Bm is obtained by multiplying each of the six matrices by its corresponding misalignment parameter and summing the results. In terms of the misalignment parameters, the particle coordinates in the misaligned reference coordinate system now take the form

$$X_f = X - Am + BXm \qquad (7)$$

To derive the forms of the matrices A and B, we consider separately the effect of each of the various misalignment parameters on each of the ray coordinates. First we derive the effect on the ray coordinates of the various misalignments as expressed in the coordinate system of the aligned magnet face. Then we will express the misalignment of the magnet face in terms of the misalignment parameters about the pivot point.

A rigid translation of the magnet face will change the x, y, and z coordinates of a ray by the amount of the displacement. The z translation will also introduce a short drift distance (positive or negative length) at the magnet face, and will contribute to B via the transformation matrix of that drift space.

To determine the effect of a rotational misalignment we form from the ray angles x' (= dx/dz) and y' (= dy/dz) and the number 1 (= dz/dz), a three-vector (x',y',1) giving the ray direction. We let $\bar{\theta}_x$, $\bar{\theta}_y$, $\bar{\theta}_z$ be the three rotational components of the misalignment vector. Then, including only first-order effects, this three-vector is transformed as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}_f = \begin{pmatrix} 1 & \bar{\theta}_z & -\bar{\theta}_y \\ -\bar{\theta}_z & 1 & \bar{\theta}_x \\ \bar{\theta}_y & -\bar{\theta}_x & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \qquad (8)$$

In the misaligned coordinate system the ray angles become

$$x'_f = x' - \bar{\theta}_y + \bar{\theta}_z y'$$

$$y'_f = y' + \bar{\theta}_x - \bar{\theta}_z x' \qquad (9)$$

Thus coordinate rotations about the aligned magnet face x and y axes only shift the ray angles. A rotation about the z axis mixes x' and y'.

If we let $\bar{m}$ represent the misalignment parameters relative to the aligned magnet face coordinate system, and $\bar{A}$ and $\bar{B}$ be the corresponding matrices, then equation (6) holds using the barred quantities. Using the results derived above, the matrices $\bar{A}$ and $\bar{B}$ are now given by

$$\bar{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (10)$$

$$\bar{B}_{125} = \bar{B}_{345} = 1$$

$$\bar{B}_{136} = \bar{B}_{246} = -\bar{B}_{316} = -\bar{B}_{426} = -1 \qquad (11)$$

The first two indices for B correspond to the ray coordinates and the third corresponds to the misalignment parameters. All other elements of $\bar{B}$ are zero.

In order to express the quantities $\bar{m}$ in terms of m, the misalignment parameters at the pivot point, we need two items. First is the orthogonal matrix O giving the three translational coordinates at the magnet face in terms of those at the pivot point

$$X_f = O\ X_p \tag{12}$$

Also needed is the three-vector P which gives the position of the origin of the aligned magnet face coordinate system, in the coordinate system of the pivot.

We now define two three-vectors which give the translational m_x and rotational m_θ parts of the misalignment vector m. We also do the same for $\bar{m}$. Then the contribution of m_x to $\bar{m}_x$ is given by equation (11), so that

$$\bar{m}_x = O\ m_x \tag{13}$$

The contribution of m_x to $\bar{m}_\theta$ is zero, since parallel translations do not affect angles.

The displacement of a point due to a rotation about the pivot is given by the vector product of the rotation vector and the position vector of the point. Therefore the displacements of the magnet face $\bar{m}_x$ due to a rotation at the pivot are given by

$$\bar{m}_x = O\left(m_\theta \times P\right) \tag{14}$$

The orthogonal transformation indicated by the matrix O gives the misalignment parameters in the magnet face coordinate system. Finally the transformation of rotational misalignment parameters is again given by equation (11), so that

$$\bar{m}_\theta = O\, m_\theta \qquad (15)$$

V. Evaluation of the Relevant Matrices

We choose the pivot to be the origin of the aligned magnet entrance face coordinate system. Therefore we have

$$A_o = \bar{A}$$

$$B_o = \bar{B} \qquad (16)$$

For the exit face, the matrix O transforms from the aligned entrance face coordinate system to the aligned exit face coordinate system. The vector P gives the position of the origin of the aligned exit face coordinate system in the aligned entrance face coordinate system. In figure 1 it is the vector which reaches from A to B.

For the exit face of a bending magnet we therefore have

$$O = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \qquad (17)$$

$$P = \begin{pmatrix} -\rho(1-\cos\alpha) \\ 0 \\ \rho\sin\alpha \end{pmatrix} \qquad (18)$$

where ρ is the radius of curvature of the central trajectory and α is the total bend angle. We then derive for the matrices A_1 and B_1

$$A_1 = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin\alpha & 1 & 0 & 0 & -\rho(1-\cos\alpha) \\ 0 & -\cos\alpha & 0 & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & -\rho(1-\cos\alpha) & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (19)$$

$$B(1)_{121} = B(1)_{341} = -\sin\alpha$$

$$B(1)_{124} = B(1)_{344} = -\rho(1-\cos\alpha)$$

$$B(1)_{125} = B(1)_{345} = \cos\alpha$$

$$B(1)_{132} = B(1)_{242} = -B(1)_{312} = -B(1)_{422} = \sin\alpha$$

$$B(1)_{136} = B(1)_{246} = -B(1)_{316} = -B(1)_{426} = -\cos\alpha \quad (20)$$

All other elements of B(1) are zero.

To calculate A_1 and B_1 for a quadrupole, we take the limit $\alpha \to 0$ with $\alpha\rho = L$, the length of the magnet being held fixed. Then we have

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -L & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (21)$$

and $B_1 = \bar{B}$ as given above.

VI. Effect on the Beam Envelope

To first order, the coordinates at the misaligned magnet exit face are related to those at the misaligned entrance face by a transfer matrix, so that

$$X_f(1) = R\,X_f(0) \qquad (22)$$

or

$$X(1) - A_1 m + B_1 X(1) m = R\left[X(0) - A_o m + B_o X(0) m\right] \qquad (23)$$

If we solve for X(1) and discard all terms in m of order higher than first, we then derive

$$X(1) = R\,X(0) + \left[A_1 - RA_o\right] m + \left[RB_o - B_1 R\right] X(0) m \qquad (24)$$

For later use we define two new matrices F and G given by

$$F = A_1 - RA_o$$

$$G = RB_o - B_1 R \qquad (25)$$

so that

$$X(1) = RX(0) + Fm + GX(0)m \tag{26}$$

An ensemble of particles in a beam line is often represented as a six-dimensional ellipsoid. The equation of this ellipsoid may be written in matrix form as follows:

$$X^T\sigma^{-1}X = 1 \tag{27}$$

where X^T is the transpose of the coordinate vector X, and σ is a real, positive definite, symmetric matrix. The square roots of the diagonal terms of the sigma matrix are a measure of the beam size in each coordinate. If the centroid of this ellipsoid does not fall on the central trajectory, then one needs to specify this centroid position also. The sigma matrix then gives the beam dimensions as measured about the centroid.

The beam envelope entering a misaligned magnet may be described in terms of the position in the aligned coordinate system of the beam centroid and the sigma matrix. For a known misalignment m, the centroid is transformed as in equation (25). The sigma matrix is transformed by

$$\sigma(1) = R\sigma(0)R^T + G\sigma(0)mR^T + R\sigma(0)m^TG^T + G\sigma(0)mm^TG^T \tag{28}$$

where the superscript T indicates a transpose.

For an uncertainty in position we define a covariance matrix $\langle mm^T \rangle$, measuring the distribution of possible magnet positions. The sigma matrix, which represents the beam envelope entering the magnet may contain contributions from both the original beam and from the uncertainty in positions of previous magnets. We assume there is no correlation of errors of positioning between any two magnets. The beam centroid is unaffected by an uncertainty in position. The transformed sigma matrix becomes

$$\sigma(1) = R\sigma(0)R^T + F\langle mm^T \rangle F^T + G\sigma(0)\langle mm^T \rangle G^T \qquad (29)$$

If the original sigma matrix is zero, then the resultant sigma matrix represents the uncertainty in the beam centroid upon leaving the magnet. If the original sigma matrix encloses a region of phase space, then the resultant sigma matrix represents the envelope of possible particle trajectories, including both the undisturbed sigma matrix and the effects of the misalignment.

VII. Implementation

This model for misalignments has been implemented in the computer program TRANSPORT.[1] An arbitrary misalignment m may be imposed on any magnet or section of the beam line. Misalignments may also be nested. The effect of all misalignments may then be added into the sigma matrix and thereby be traced through the system. Alternatively, the effects of separate components of the misalignment vector on individual magnets may be stored in a table. This table is traced through the beam line and may be compared with the unperturbed sigma matrix at any later point. Details of implementation are described in the TRANSPORT manual.

Figure Captions

Figure 1 Perfectly aligned and misaligned bending magnets. With the misaligned magnet the beam line reference trajectory is no longer continuous with that inside the magnet. The displacements of the origins of the entrance and exit face reference coordinate systems are shown as $\bar{D}_o$ and $\bar{D}_1$ respectively.

Figure 2 Magnet entrance and exit face coordinate systems. The misalignment causes both a translation and a rotation of the reference coordinate system.

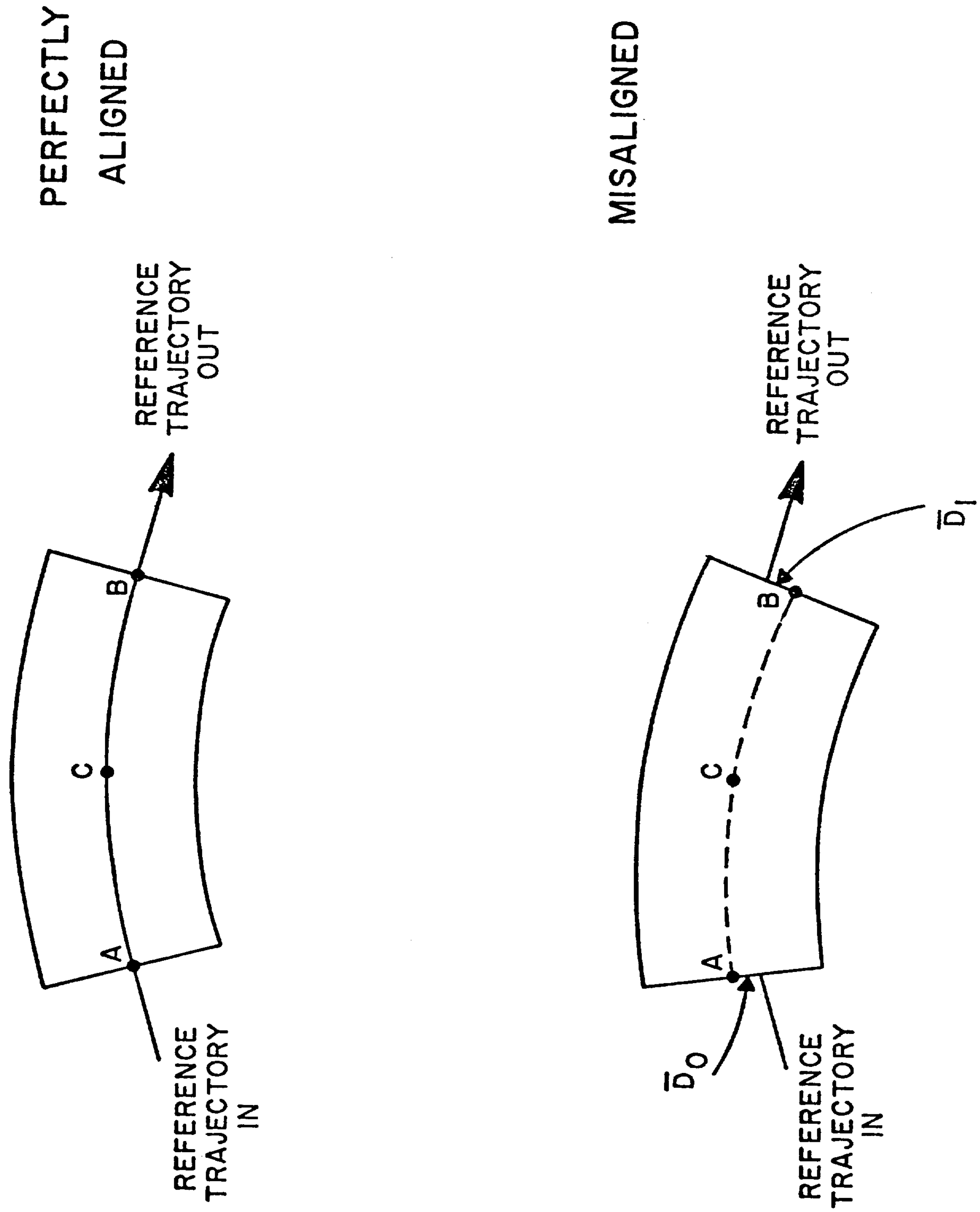
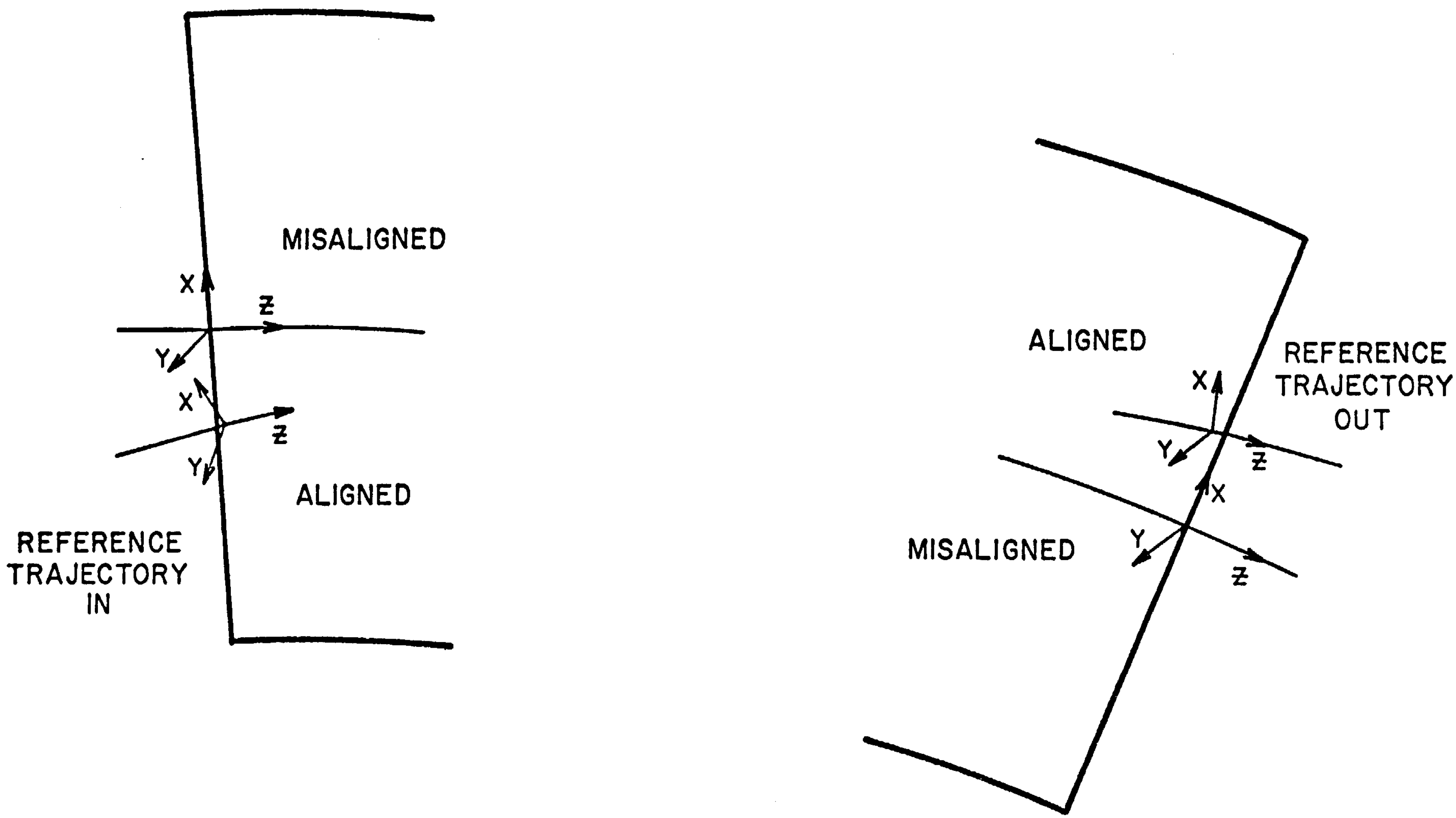

PERFECTLY ALIGNED
REFERENCE TRAJECTORY IN
A
C
B
REFERENCE TRAJECTORY OUT
MISALIGNED
$\bar{D}_O$
REFERENCE TRAJECTORY IN
A
C
B
$\bar{D}_I$
REFERENCE TRAJECTORY OUT

figure 1



MAGNET ENTRANCE FACE COORDINATE SYSTEM

MAGNET EXIT FACE COORDINATE SYSTEM

figure 2

FIRST-ORDER PARAMETER OPTIMIZATION AND COVARIANCE

Section V

To optimize the selected parameters, TRANSPORT uses the method of non-linear least squares, differential correction, a good description of which is found in Ref. (7) below pages, 390-393. A useful by-product of this method is the covariance matrix C, printed by the program at the successful conclusion of any run involving parameter fitting. In many applications C may be used to estimate tolerances on the fitted parameters subject to the specified tolerances (i.e., the standard deviations) of the constraints.

The covariance matrix C is symmetric. This admits a geometrical interpretation as an ellipsoid, and is printed in the same suggestive format as is the beam ellipsoid σ, except that in this case the dimension is equal to N, the number of parameters varied. The center of the ellipsoid is at

$$\vec{\lambda^o} = (\lambda_1^o , \lambda_2^o \; . \; . \; . \; \lambda_N^o) \quad , \tag{1}$$

the N values found by TRANSPORT to be the best estimate of the varied parameters. The equation of the covariance ellipsoid is

$$(\vec{\lambda} - \vec{\lambda^o}) \; C^{-1} \; (\vec{\lambda} - \vec{\lambda^o})^T \leq 1 \tag{2}$$

where $(C_{11})^{1/2}$, the first diagonal element printed, measures the maximum extent of the ellipsoid along the λ_1 axis (the first varied parameter) in the same sense that $(\sigma_{11})^{1/2}$ measures the maximum extent of the beam ellipsoid along the x axis. The off diagonal elements are normalized so that they are ≤ 1 in magnitude, in analogy with the r_{ij} of the beam matrix, and can be interpreted as measures of the orientation of the covariance ellipsoid.

(7) SOLMITZ, Analysis of Experiments in Particle Physics, Ann. Rev. Nuc. Sci., Vol 14, 1964.

K. Halbach, "A Program for Inversion of System Analysis and its Application to the Design of Magnets", Second International Conference on Magnet Technology, Oxford, 1967.

The best estimates (or optimized values) of the varied parameters λ are precisely those that minimize the quantity:

$$\chi^2 = \sum_{K=1}^{M} \left[\frac{\xi_K - f_K(\lambda_1 \ldots\ldots \lambda_N)}{S_{KK}} \right]^2 \qquad (3)$$

where:

M = number of contraints

f_K = a function selected by the code digits (i,j) on the constraint definition (Type Code 10.). For example, i = -1, j = 1 means that the transform matrix element R_{11} is to be constrained.

ξ_K = the desired value of the selected function.

S_{KK} = the desired accuracy of fit (i.e., the standard deviation).

In our notation this minimum is expressed by:

$$\chi^2_{min} = \sum_{K=1}^{M} \left[\frac{\xi_K - f_K(\lambda_1^o \ldots\ldots \lambda_N^o)}{S_{KK}} \right]^2 \qquad (4)$$

χ^2_{min} is printed at the successful conclusion of any run involving parameter fitting. Whether or not the optimization (χ^2_{min}, $\vec{\lambda^o}$) is 'acceptable' depends on each application and must be evaluated by the user. Values of $\chi^2_{min} \sim (M-N)$ are sometimes (but not necessarily) regarded as 'good'. In particular if M=N, then an exact solution, $\chi^2_{min} = 0$ should be found by TRANSPORT.

If the resulting fit is acceptable, then the following interpretation may be put on the covariance matrix C: Let the parameters be changed to values $\vec{\lambda}$ near the optimum $\vec{\lambda^o}$, such that they stay within the ellipsoid defined by:

$$(\vec{\lambda} - \vec{\lambda^o})\ C^{-1}\ (\vec{\lambda} - \vec{\lambda^o})^T \leqslant 1$$

Then the resulting deviation of the specified constrained quantities is bounded by:

$$\sum_{K=1}^{M} \left[\frac{\xi_K - f_K(\lambda)}{S_{KK}} \right]^2 \leqslant \chi^2_{min} + 1 \qquad (5)$$

This interpretation is strictly true if the constrained functions f_K are linear in the parameters $\vec{\lambda}$. In the non-linear case, it is an approximation valid only in some neighborhood of $\vec{\lambda^o}$.

Example:

On the following page is an example of a TRANSPORT data deck and the resulting covariance fit of a first-order run. We have ask for a point-to-point image in both the x and y planes by varying the fields of the quadrupole triplet. The following definitions and solution are applicable:

$$B_1 = \lambda_1,\ B_2 = \lambda_2,\ B_1^o = \lambda_1^o,\ B_2^o = \lambda_2^o,\ f_1 = R_{12},\ f_2 = R_{34},\ \xi_1 = 0,\ \xi_2 = 0$$

$$S_{11} = 0.005,\ S_{22} = 0.005 \text{ and } N=2.$$

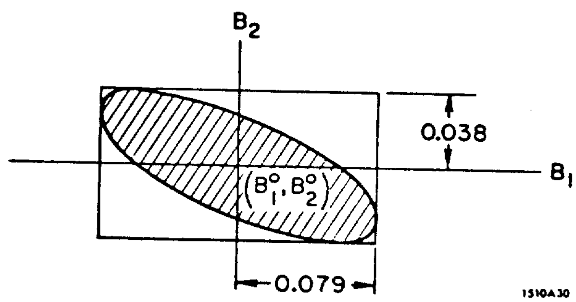
From the data deck and the TRANSPORT printout* shown on the following page, we learn that:

$$\frac{\chi^2_{min}}{N} = 0.8$$

$$B_1^o = -\ 7.4096 \text{ Kg},\ B_2^o = 6.1577 \text{ Kg}$$

$$\left.\begin{array}{l}\sqrt{C_{11}} = 0.079 \\ r_{12} = -\ 0.883 \quad \sqrt{C_{22}} = 0.038\end{array}\right\} \text{Covariance matrix information}$$

The ellipsoid $\overrightarrow{(B - B^o)}\ C^{-1}\ \overrightarrow{(B - B^o)}^T \leqslant 1.0$ can then be constructed as shown:



*Note: the printout format is:

$$\begin{array}{llll}\text{COVARIANCE (FIT} & \chi^2_{min}/N) & & \\ \sqrt{C_{11}} & & & \\ r_{12} & \sqrt{C_{22}} & & \\ \vdots & \vdots & \ddots & \\ r_{1n} & \cdots\cdots & r_{n,n-1} & \sqrt{C_{NN}}\end{array}$$

Example:

```
'USE TRIPLET TO FORM POINT-TO-POINT IMAGE AT TARGET'
0
13. 2. ;
15. 1. 'IN' ;  15. 8. 'FT' ;
1. 0. 0. 0. 0. 0. 0. 1.05 ;
3.0 12. ;
5.02 1.5 -8.0 4.0 'Q1' ;
3.0 0.5 ;                                   } Data Deck for TRANSPORT Run
5.01 3.0 7.0 4.0 'Q2' ;
3.0 0.5 ;
5.02 1.5 -8. 4.0 'Q1' ;
3.0 7.0 ;
10. -1. 2. 0. .005 'FIT1' ;
10. -3. 4. 0. .005 'FIT2' ;
SENTINEL
/*
```

(χ^2_{min})

```
*COVARIANCE  ( FIT   0.8 )
 0.079          } Covariance Matrix
-0.883  0.038   }
USE TRIPLET TO FORM POINT-TO-POINT IMAGE AT TARGET

*BEAM*     1.000000     1.05 GEV

*DRIFT*    3.0     12.0000 FT     B1°

*QUAD*     5.00    1.50000 FT   -7.4096 KG    4.000 IN   (  -3.208 FT )

*DRIFT*    3.0      0.5000 FT     B2°

*QUAD*     5.00    3.00000 FT    6.1577 KG    4.000 IN   (   2.673 FT )

*DRIFT*    3.0      0.5000 FT     B1°

*QUAD*     5.00    1.50000 FT   -7.4096 KG    4.000 IN   (  -3.208 FT )

*DRIFT*    3.0      7.0000 FT
                                 ξ1
*FIT*     10.0     -1.  2.      0.0     / 0.005    { Constraint on R12 }
                               -0.000
        LABEL = FIT1
                                 ξ2
*FIT*     10.0     -3.  4.      0.0     / 0.005    { Constraint on R34 }
                               -0.000
        LABEL = FIT2
 *LENGTH*    26.0000 FT
```

Interpretation:

So long as B_1 and B_2 fall inside the shaded area (this is the tolerance requirement), then the ellipsoid representing the corresponding deviations of the matrix elements R_{12} and R_{34} is using Eq. (5):

$$\left(\frac{0 - R_{12}}{0.005}\right)^2 + \left(\frac{0 - R_{34}}{0.005}\right)^2 \leqslant \chi^2_{min} + 1.0 = 2.6$$

or

$$R_{12}^2 + R_{34}^2 \leqslant 2.6\,(0.005)^2$$

Note that it is not enough to prescribe tolerances $|\Delta B_1| < .079$ and $|\Delta B_2| < .038$, since there is unshaded area inside the rectangle defined by these values. The strongly tilted covariance ellipse (i.e., $|r_{12}| \sim 1$) suggests that the triplet power supplies should be designed so that any drift in magnetic field B_1 causes a compensating drift in the magnetic field B_2 so as to stay inside the shaded area shown in Figure 1.