U. R.

SCHWEIZERISCHES INSTITUT FÜR NUKLEARFORSCHUNG
INSTITUT SUISSE DE RECHERCHES NUCLÉAIRES
ISTITUTO SVIZZERO DI RICERCHE NUCLEARI
SWISS INSTITUTE FOR NUCLEAR RESEARCH

REPRESENTATION OF BEAM ELLIPSES FOR TRANSPORT CALCULATIONS

W. JOHO

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TITEL:	NAME: W. Joho /KIP	*finished
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Introduction

It is customary to represent a particle beam (moving in the z-direction) as a 6-dimensional ellipsoid in phase space. If this ellipsoid is projected into a twodimensional subspace (x, x') $(x'=\frac{dx}{dz})$ the result is an ellipse in this phase space (x, x') with area $\pi \cdot \varepsilon$. Different representations for this beam ellipse are listed. Some interesting properties of so called binomial phase-space distributions are given. For emittance measurements it is proposed to plot ln(p) versus emittance, where p is the fraction of particles outside a given emittance.

I. Courant-Snyder representation[1] (α, β, γ, ε)

The ellipse is characterized with the Twiss parameter α, β, γ and the emittance parameter ε (area = $\pi \cdot \varepsilon$). The ellipse equation is

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon} \qquad \text{(Courant-Snyder)} \qquad (1)$$

The Twiss parameters α, β, γ are correlated:

$$\boxed{\beta\gamma - \alpha^2 = 1} \qquad (2)$$

$$\alpha = -\frac{1}{2}\beta' \qquad (3)$$

The dimensions are: $[\alpha] = 1$

$[\beta] = m$ always positive

$[\gamma] = m^{-1}$ always positive

Sign of α: for a divergent beam (as in figure 1) α is negative.

Transformation of the Twiss parameters through an arbitrary beam transfer section:

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = M \begin{pmatrix} x_i \\ x'_i \end{pmatrix} \qquad (4)$$

where M is the transfer matrix between initial point i and final point f:

$$M \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_f = M \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_i M^T$$

which can be written as:

$$\begin{pmatrix} \beta_f \\ \alpha_f \\ \gamma_f \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}\,m_{12} & m_{12}^2 \\ -m_{21}m_{11} & 1+2m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_i \\ \alpha_i \\ \gamma_i \end{pmatrix} \quad (6)$$

A related parameter is the so called phase advance angle μ

$$\mu \equiv \int \frac{dz}{\beta(z)}$$

II. Parametric representation (x_m, θ_m, χ)

The ellipse is characterized with the three parameters

x_m = maximum value of x

θ_m = maximum value of x'

χ = correlation phase for orientation of ellipse

For a divergent beam (as in fig. 1) χ is positive

The connection with the notation r_{12} of TRANSPORT[2] is:

$$\boxed{\sin\chi \equiv r_{12}} \quad (7)$$

A point on the ellipse is given by

$$\boxed{\begin{aligned} x &\equiv x_m \cos\delta \\ x' &\equiv \theta_m \sin(\delta+\chi) \end{aligned}} \quad (8)$$

where δ is a running parameter between 0 and 2π. The area of the ellipse is $\pi\varepsilon$ with

$$\boxed{\varepsilon = x_m \, \theta_m \cos\chi} \tag{9}$$

The meaning of the ellipse parameters is best illustrated with the coordinates of the four specific points 1, 2, 3, 4 on the ellipse. Points 1 and 3 as well as points 2 and 4 are conjugate pairs, i.e. the tangent at point 1 is parallel to line 0-3, the tangent at point 4 is parallel to line 0-2. Notice the nice symmetry of the coordinates in the parametric representation. The slope of the envellope $x_m(z)$ is given by x_2' .

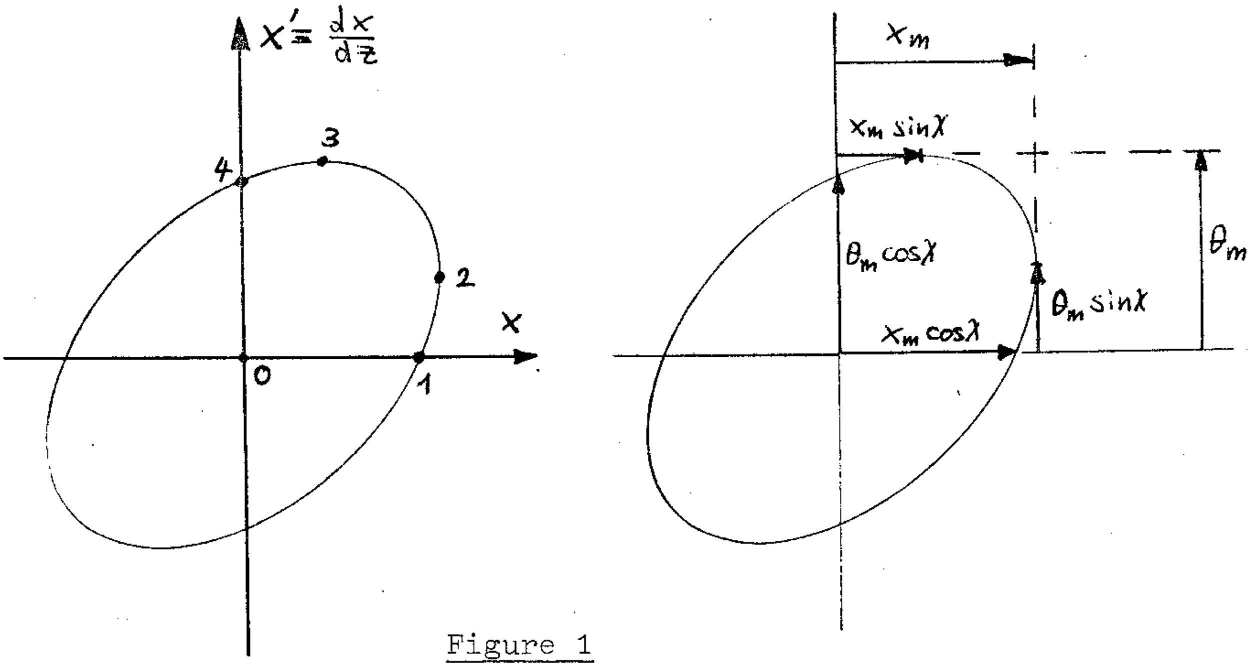


Figure 1

Parametric: $\tan\chi \equiv \frac{x_2'}{x_4'} = \frac{x_3}{x_1} = -\alpha$

Courant-Snyder: $\alpha \equiv -\frac{x_2'}{x_4'} = -\frac{x_3}{x_1}$, $\beta = \frac{x_2}{x_4'}$

$\gamma = \frac{x_3'}{x_1}$

$\varepsilon = x_2 \cdot x_4' = x_1 \cdot x_3'$

point	δ	x (Parametric)	x' (Parametric)	x (Courant-Snyder)	x' (Courant-Snyder)
1)	$-\chi$	$x_m \cos\chi$	0	$\sqrt{\frac{\varepsilon}{\gamma}}$	0
2)	0	x_m	$\theta_m \sin\chi$	$\sqrt{\varepsilon\beta}$	$-\alpha\sqrt{\frac{\varepsilon}{\beta}}$
3)	$90^0-\chi$	$x_m \sin\chi$	θ_m	$-\alpha\sqrt{\frac{\varepsilon}{\gamma}}$	$\sqrt{\varepsilon\gamma}$
4)	90^0	0	$\theta_m \cos\chi$	0	$\sqrt{\frac{\varepsilon}{\beta}}$

The connection between the two representations is:

$$x_m = \sqrt{\varepsilon\beta} \qquad \beta = \frac{x_m}{\theta_m \cos\chi} = \frac{x_m^2}{\varepsilon}$$

$$\theta_m = \sqrt{\varepsilon\gamma} \qquad \gamma = \frac{\theta_m}{x_m \cos\chi} = \frac{\theta_m^2}{\varepsilon} \qquad (10)$$

$$\chi = -\tan^{-1}\alpha \qquad \alpha = -\tan\chi$$

$$\left(r_{12} \equiv \sin\chi = \frac{-\alpha}{\sqrt{\beta\gamma}}\right)$$

parametric	Courant-Snyder
emittance is incorporated in parameters. Zero emittance gives no problem ($\chi = \pm 90^0$)	Parameters define only shape of ellipse, emittance is extra parameter. Singularity for $\varepsilon = 0$.
Ellipse easy to plot with angle δ as continous parameter.	Cumbersome to plot ellipse: for each position x a quadratic equation gives x'. Points are not evenly distributed along ellipse.
This representation is easily extended to more than two dimensions with correlations χ_{ij} between x_i and x_j.	nice transformation properties of Twiss parameters α, β, γ (see eq. 6)
Transformation through α, β, γ	

III. Representation of ellipse with complex number Y

The emittance area $\pi\varepsilon$ is a free parameter. Two more parameters (= one complex number) are thus needed for the shape of the ellipse.

In a drift we have:

$x_1 \equiv x_w$ = const. = amplitude at waist

$x_3' = \theta_m$ = const. = maximum divergence

Define therefore:

$$R \equiv \frac{x_1}{x_3'} = \frac{1}{\gamma} = \frac{x_m}{\theta_m} \cos\chi \quad = \text{ ratio of main axis at waist}$$

(11)

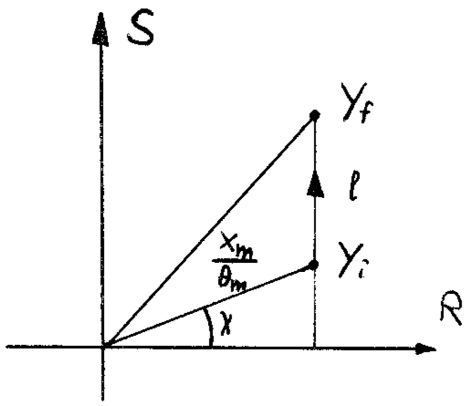
$$S \equiv \frac{x_3}{x_3'} = -\frac{\alpha}{\gamma} = \frac{x_m}{\theta_m} \sin\chi \quad = \text{ distance of beam from waist position}$$

This form of R and S suggests a complex number[3,4]

$$\boxed{Y \equiv R + iS = \frac{x_m}{\theta_m} e^{i\chi}} \qquad [\text{dimensions} = m] \qquad (12)$$

The transformation of Y over a drift length ℓ is given by:

$R_f = R_i$; $S_f = S_i + \ell$ or



$$\boxed{Y_f = Y_i + i\ell} \qquad (13)$$

Figure 2

The inverse of Y has also interesting properties:

$$Y^{-1} \equiv U + iV = \frac{\theta_m}{x_m} e^{-i\chi} \qquad [\text{dimensions} = m^{-1}]$$

$$U = \frac{\theta_m}{x_m} \cos\chi = \frac{x_4'}{x_2}$$

$$V = -\frac{\theta_m}{x_m} \sin\chi = -\frac{x_2'}{x_2}$$

A thin lens of focal power $K \equiv \frac{1}{f}$ acts like a drift in the x'-direction:

$$x_4' = \text{const.}$$

$$x_2 = \text{const.} = x_m$$

$$x_{2f}' = x_{2i}' - Kx_{2i}$$

which leads to $U_f = U_i = \text{const.}$

$V_f = V_i + K$ or

$$\boxed{Y_f^{-1} = Y_i^{-1} + iK} \quad = \text{transformation through thin lens} \quad (16)$$

A succession of drifts and thin lenses has therefore simple transformation properties.

IV. Representation in Program TRANSPORT

ellipse equation:

$$\boxed{\vec{x}^T \sigma^{-1} \vec{x} = 1} \qquad \vec{x} = \binom{x}{x'} \qquad (17)$$

or explicitly $\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2 = \varepsilon^2$

with

$$\sigma = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

$$\sigma^{-1} = \frac{1}{\varepsilon} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

$$\begin{aligned} \sigma_{11} &= \varepsilon\beta = x_m^2 \\ \sigma_{22} &= \varepsilon\gamma = \theta_m^2 \\ \sigma_{12} &= -\varepsilon\alpha = x_m \theta_m \sin\chi = \varepsilon \tan\chi \end{aligned} \qquad (18)$$

$$\left(r_{12} = \sin\chi = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\,\sigma_{22}}}\right)$$

$$\det(\sigma) = \varepsilon^2$$

With a transfer matrix R the σ-matrix transfers as follows:

$$\boxed{\vec{x}_f = R\,\vec{x}_i} \qquad (19)$$

$$\boxed{\sigma_f = R\,\sigma_i\,R^T} \qquad (20)$$

V. Transformation of ellipse parameters in drift space

We assume that the beam has a waist at position Z_W with a minimum amplitude x_W.

The distance from the waist was defined in (11) as

$$S = Z - Z_W = \frac{x_3}{x_3'} \qquad \text{(positive for divergent beam)}$$

This is the same definition as in the CERN report[5], but has the opposite sign as the notation in the TRANSPORT - report[2].

The beam ellipse transforms in a drift as follows:

<u>parametric:</u>

$$\theta_m = x_3' = \text{const.}$$

$$x_m^2 = x_W^2 + \theta_m^2\,(Z-Z_W)^2 = \text{envellope equation} = \text{hyperbola}$$

$$\tan\chi = \frac{\theta_m}{x_W}\,(Z-Z_W) = \frac{Z-Z_W}{\beta_W} \qquad \left(\text{or } \cos\chi = \frac{x_W}{x_m}\right) \qquad (21)$$

$$\chi' \equiv \frac{d\chi}{dZ} = \frac{\theta_m}{x_W}\cos^2\chi \quad \left(= \frac{1}{\beta(Z)}\right) \;\rightarrow\; \chi = \text{phaseadvance from waist!}$$

$$x_m' \equiv \frac{dx_m}{dZ} = x_2' = \theta_m \sin\chi = \text{slope of envellope}$$

Complex representation $Y \equiv R + iS$

As seen in (13) we have

$$R = \text{const.} = \beta_w = \frac{x_w}{\theta_m} \qquad S = Z - Z_w \tag{22}$$

Courant-Snyder

$$\begin{aligned} \gamma &= \text{const.} \\ \beta &= \beta_i - 2\alpha_i (Z-Z_i) + \gamma_i (Z-Z_i)^2 \\ \alpha &= \alpha_i - \gamma_i (Z-Z_i) \end{aligned} \tag{23}$$

or with $\beta = \beta_w$ at waist:

$$\begin{aligned} \beta &= \beta_w + \frac{1}{\beta_w} (Z-Z_w)^2 \qquad \boxed{\beta_w = \frac{x_w}{\theta_w}} \\ \alpha &= \frac{1}{\beta_w} (Z-Z_w) \\ \beta' &= -2\alpha \end{aligned} \tag{24}$$

The equation for β shows an easy interpretation of β_w: At a distance β_w from the waist ($Z = Z_w + \beta_w$) the parameter β has doubled to $2\beta_w$. Therefore the beam amplitude x_m has increased from x_w to $\sqrt{2}\, x_w$, independent of ε.

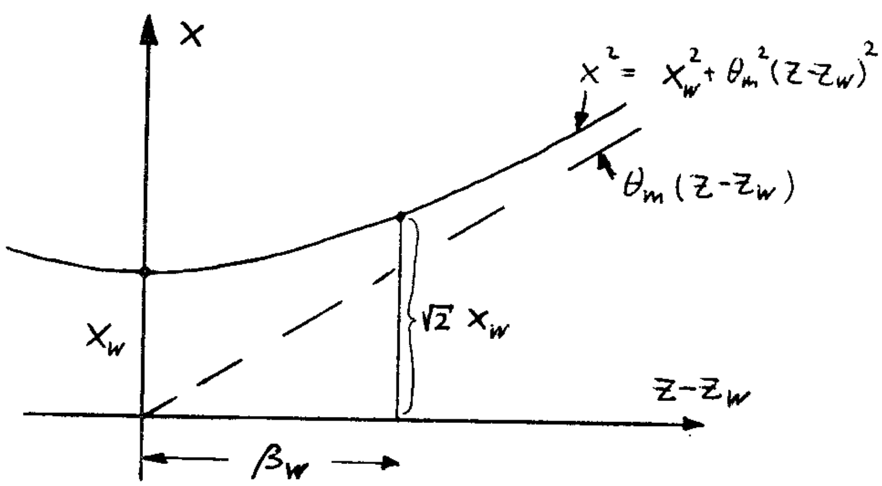


Figure 3

The phase advance μ is (assuming a waist at z=0):

$$\mu \equiv \int_0^Z \frac{dZ}{\beta(Z)} = \beta_W \int_0^Z \frac{dz}{\beta_W^2 + Z^2}$$

$$\boxed{\mu = \tan^{-1} \frac{Z}{\beta_W} = \tan^{-1} \frac{\theta_m}{x_W} Z} \qquad (25)$$

$$\mu_\infty = \frac{\pi}{2} \qquad (26)$$

VI. Normalized phase space coordinates

transformation of ellipse to circle of radius $\sqrt{\varepsilon}$:

ellipse:

$$\gamma x^2 + 2\alpha\, xx' + \beta x'^2 = \varepsilon$$

this can be written as

$$\frac{x^2 + (\alpha x + \beta x')^2}{\beta} = \varepsilon \qquad (27)$$

this suggests a coordinate transformation to new normalized variables η, η':

$$\boxed{\eta \equiv \frac{x}{\sqrt{\beta}} \qquad \eta' \equiv \frac{\alpha}{\sqrt{\beta}} x + \sqrt{\beta} x'} \qquad (28)$$

dimension of η, $\eta' = m^{1/2}$

the ellipse equation becomes now

$$\boxed{\eta^2 + \eta'^2 = \varepsilon} \qquad (29)$$

= circle with radius $\sqrt{\varepsilon}$ and area $\pi \cdot \varepsilon$

the transformation can also be written as:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \qquad (30)$$

or in the parametric representation:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\varepsilon}}{x_m} & 0 \\ \frac{-\theta_m \sin\chi}{\sqrt{\varepsilon}} & \frac{x_m}{\sqrt{\varepsilon}} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \frac{1}{\sqrt{\varepsilon}} \begin{pmatrix} \theta_m \cos\chi & 0 \\ -\theta_m \sin\chi & x_m \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \qquad (31)$$

VII. Definition of beam ellipse for arbitrary distribution of particles

For an arbitrary distribution of particles it is important to define exactly what we mean by beam ellipse. For this purpose one works with the statistical parameters :

$$\overline{x^2} \equiv \sigma^2 = \iint x^2 \rho(x,x')\,dx\,dx'$$

$$\overline{x'^2} \equiv \sigma'^2 = \iint x'^2 \rho(x,x')\,dx\,dx' \qquad (32)$$

$$\overline{xx'} = \iint xx' \rho(x,x')\,dx\,dx'$$

where $\rho(x,x')$ is the particle density in phasespace (x,x') with the normalisation

$$\iint \rho(x,x')\,dx\,dx' = 1 \qquad (33)$$

For a finite number N of "superparticles" we have the usual definition

$$\overline{x^2} = \frac{1}{N} \sum_{i=1}^{N} x_i^{\,2} \qquad (34)$$

and analogues for the other parameter.

It is now useful to define as RMS (=root mean square) beam ellipse parameter x_m, Θ_m, χ and ε in the parametric representation:

$$\left.\begin{aligned} x_m &\equiv 2\,\sigma \\ \Theta_m &\equiv 2\,\sigma' \\ r_{12} &\equiv \sin\chi \equiv \frac{\overline{xx'}}{\sigma\sigma'} = \frac{4\overline{xx'}}{x_m\Theta_m} \\ \varepsilon &= x_m\,\Theta_m \cos\chi = 4\sqrt{\overline{x^2}\,\overline{x'^2} - \overline{xx'}^2} \end{aligned}\right| \qquad (35)$$

The factor 2 in the definitions above is arbitrary,[6,7] but has the convenience that x_m, Θ_m correspond exactly to the maximum values of x, x' for an ellipse with uniform phase space density (see appendix). The beam ellipse is then defined with the equations

$$\begin{aligned} x &= x_m \cos\delta \\ x' &= \Theta_m \sin(\delta+\chi) \end{aligned} \qquad (0 \leq \delta \leq 2\pi) \qquad (36)$$

with the usual emittance $\pi\cdot\varepsilon$. With the above definition of ε about 86-88 % of the total beam are contained within this beam ellipse for reasonably behaved beam profiles generally encountered in practice. The Twiss parameter α,β,γ have also simple meanings[8]:

$$\beta = \frac{\overline{x^2}}{E} \quad , \quad \gamma = \frac{\overline{x'^2}}{E} \quad , \quad \alpha = -\frac{\overline{xx'}}{E}$$

$$E^2 \equiv \overline{x^2} \cdot \overline{x'^2} - \overline{xx'}^2 \quad , \quad (\varepsilon = 4E) \qquad (37)$$

The RMS-parameter (35) are normally used in beam envellope calculations involving space charge forces[6,9] and x_m is simply called the x-amplitude.

VIII. Determination of beam ellipse from 3 beam profiles in a drift space

At a given point z the beam profile f (x) is obtained by

$$f(x) = \int \rho(x,x')dx' \qquad (38)$$

with the normalization from (33):

$$\int f(x)dx = 1 \qquad (39)$$

The amplitude x_m is defined by (32) and (35) as:

$$x_m^2 = 4 \int x^2 f(x)dx \qquad (40)$$

Lets suppose now that the beam amplitude (x) was measured at 3 positions $z=z_-$, o, z_+ along a drift space with the values x_-, $x_o \equiv x_m$ and x_+ respectively. With this measurement the other ellipse parameter Θ_m and χ (beside x_m) can be determined at the middle point z=o.

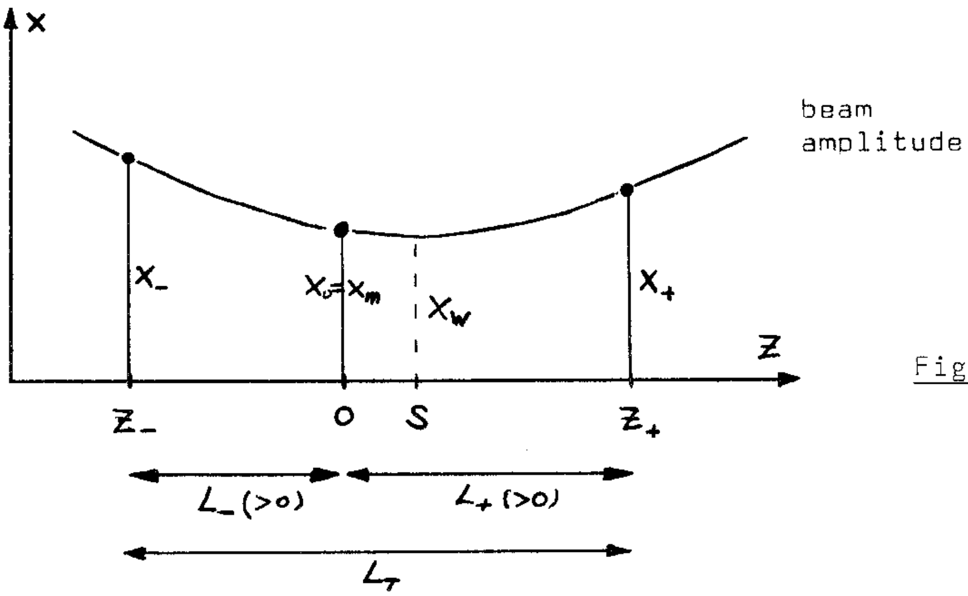


Figure 4

The beam is supposed to have a waist at z=S. The following determination of the ellipse parameter is only reliable, if this waist is close to the middle point z=o. The general ellipse equation (1) is evaluated at z=o:

$$\gamma_o x^2 + 2\alpha_o xx' + \beta_o x'^2 = \varepsilon \qquad (41)$$

evaluating the ellipse equations at z_+ and z_- and using the equations (23) and (10) for $\beta(z)$ we obtain

$$x_+^2 = x_o^2 - 2\alpha_o \varepsilon L_+ + \Theta_m^2 L_+^2$$
$$x_-^2 = x_o^2 + 2\alpha_o \varepsilon L_- + \Theta_m^2 L_-^2 \qquad (42)$$

from these 2 equations we can deduct Θ_m and $\alpha_o\varepsilon$ ($x_m = x_o$ is already known from the measurement) and we obtain

$$\left.\begin{aligned}
\theta_m^{\,2} &= \frac{\delta_+}{L_+L_T} + \frac{\delta_-}{L_-L_T} \\
\tilde{\alpha} &\equiv -\alpha_o\varepsilon = \frac{1}{2}\left(\frac{L_-}{L_+L_T}\,\delta_+ - \frac{L_+}{L_-L_T}\,\delta_-\right) \\
r_{12} &= \sin\chi = \frac{\tilde{\alpha}}{x_m\theta_m} \\
\varepsilon &= x_m\,\theta_m\,\cos\chi = \sqrt{x_m^{\,2}\,\theta_m^{\,2} - \tilde{\alpha}^2}
\end{aligned}\right\} \quad (43)$$

where we have used the abreviations

$$\delta_{\pm} \equiv x_{\pm}^{\,2} - x_o^{\,2} \qquad (44)$$

For consistent measurements $|\sin\chi|$ cannot be bigger than one. The waist $x_w = \varepsilon/\theta_m$ is at $z=S$ with $S = \frac{x_m}{\theta_m}\sin\chi$ from (11):

$$S = \frac{\tilde{\alpha}}{\theta_m^{\,2}} = \frac{1}{2}\;\frac{L_-^{\,2}\delta_+ - L_+^{\,2}\delta_-}{L_-\delta_+ + L_+\delta_-} \qquad (45)$$

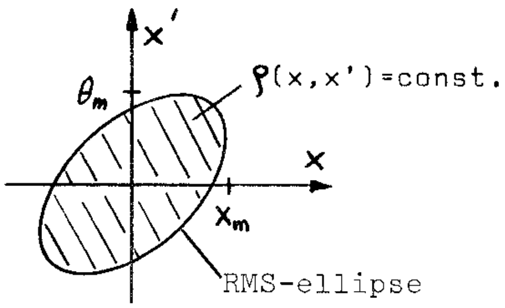
Tomographic methods have been developped[10], which allow even a reconstruction of the phase space density $\rho(x,x')$ from 3 profile measurements. In this iterative scheme the beam parameter (43) can be conveniently used for an initial guess of the particle distribution. It has been shown [11], that optimal results are obtained if the middle profile is at the waist position and the other profiles are measured at a distance $l=\sqrt{3}\cdot\beta_w = \sqrt{3}\,x_w^2/\varepsilon$ from the waist, where the beam amplitude has exactly doubled. This corresponds to three projections at 120^o intervals of a circle in normalized phase space.

APPENDIX

RMS-ellipses for some specific particle distributions

In order to get some feeling for the RMS-ellipse we show examples of typical particle distributions and their corresponding ellipses, calculated from eq. (35).

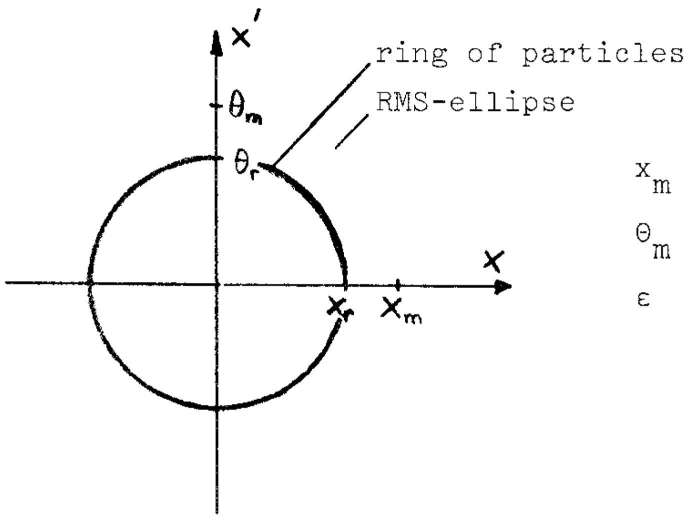
1. Uniform phase space density inside a given ellipse



The RMS-ellipse corresponds exactly to the circumscribing ellipse, due to the factor 2 in the definition of x_m and Θ_m in (35).

This density distribution is obtained by projecting a uniformly populated four-dimensional ellipsoidal shell[12] into the subspace (x,x').

2. Hollow beam, $\rho(x,x') = 0$ except on a ring



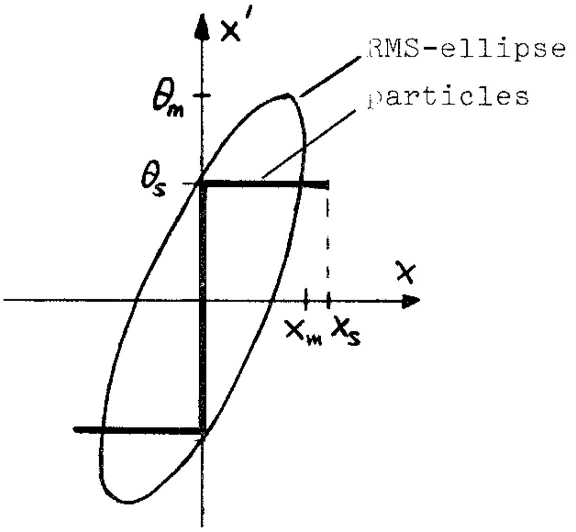
$$x_m = \sqrt{2}\, x_r$$

$$\Theta_m = \sqrt{2}\, \Theta_r$$

$$\varepsilon = x_m \Theta_m = 2\, x_r \Theta_r$$

3. S-shaped beam

Due to aberrations one can obtain beams with an S-shaped distribution in phase space. Exaggerated it could look like this:



one obtains:

$$x_m = \frac{\sqrt{6}}{3} x_s \sim .8\ x_s$$

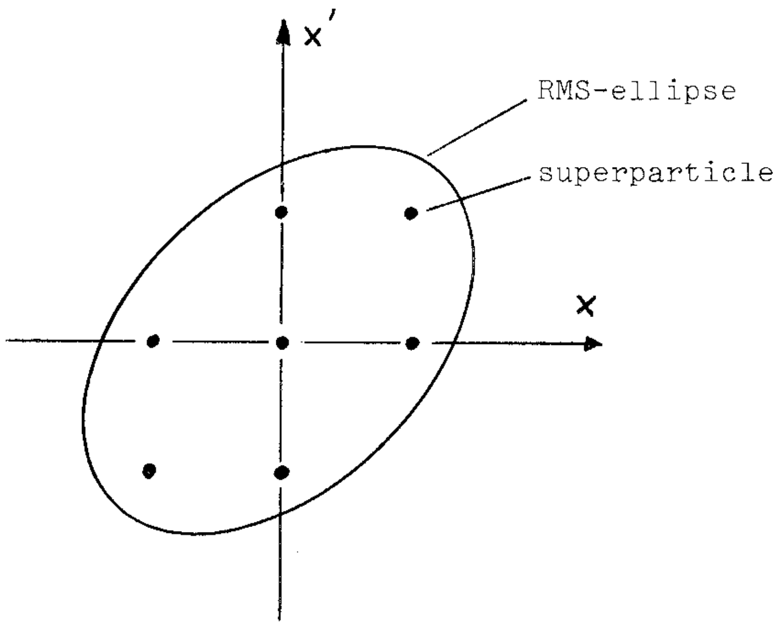
$$\Theta_m = \frac{2\sqrt{6}}{3}\Theta_s \sim 1.6\ \Theta_s$$

$$r_{12} = \sin\chi = .75$$

$$\varepsilon = x_m\Theta_m \cos\chi = \frac{\sqrt{7}}{3} x_s\Theta_s$$

In both examples 2 and 3 the RMS-emittance $\pi\varepsilon$ is not zero, although the area occupied by the beam is zero.

4. "Superparticles"



5. Two-dimensional Gaussian function

$$\rho(x,x') = \frac{1}{\sqrt{2\pi}\,\sigma}\,\frac{1}{\sqrt{2\pi}\,\sigma'}\,\exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma^2}+\frac{x'^2}{\sigma'^2}\right)\right) \qquad \text{(A.1)}$$

where we assumed that the beam is examined at a waist with no coupling term xx' present. Otherwise we can always bring ρ into the above form by a transformation à la (28).
The beam profile f(x) is given by:

$$f(x) = \int \rho(x,x')dx' = \frac{1}{\sqrt{2\pi}\,\sigma}\,\exp\left(-\frac{1}{2}\,\frac{x^2}{\sigma^2}\right) \qquad \text{(A.2)}$$

with a FWHM-value Γ of: $\Gamma = \sqrt{8\ \ln 2}\,\sigma = 2.35\ \sigma$

$$\text{normalization:} \quad \iint \rho(x,x')dxdx' = \int f(x)dx = 1 \qquad \text{(A.3)}$$

The second moments are given by:

$$\overline{x^2} = \sigma^2, \quad \overline{x'^2} = \sigma'^2 \qquad \text{(A.4)}$$

The RMS-ellipse is thus defined by

$$\begin{aligned} x_m &= 2\sigma = .85\,\Gamma \\ \Theta_m &= 2\sigma' \\ \chi &= 0 \\ \varepsilon &= x_m\Theta_m = 4\sigma\sigma' \end{aligned} \qquad \text{(A.5)}$$

The fraction q of the beam which is contained between the values $-x_m$ and $+x_m$ is given by

$$\int_{-x_m}^{x_m} f(x)dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-2\sigma}^{+2\sigma} \exp\left(-\frac{1}{2}\,\frac{x^2}{\sigma^2}\right)dx = q \qquad \text{(A.6)}$$

From integration tables we obtain the result q = .954

It is also interesting to know what fraction p of the beam is lying <u>outside</u> an ellipse of area $\pi\varepsilon_p = \pi\cdot\sigma\sigma' a^2$ and given by

$$\frac{x^2}{\sigma^2} + \frac{x'^2}{\sigma'^2} = a^2 \qquad \text{(A.7)}$$

For this purpose we have to solve the integral

$$\frac{1}{2\pi\sigma\sigma'} \iint \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma^2} + \frac{x'^2}{\sigma'^2}\right)\right) dx\,dx' = 1-p \qquad \text{(A.8)}$$

The result is

$$p = \exp\left(-\frac{\varepsilon_p}{2\sigma\sigma'}\right) = \exp\left(-\frac{2\varepsilon_p}{\varepsilon}\right) \qquad \text{(A.9)}$$

putting $\varepsilon_p = \varepsilon$ gives p = .14, which means that for a two-dimensional Gaussian distribution 86 % of the beam is contained in the RMS-ellipse of area $\pi\varepsilon = \pi\cdot 4\sigma\sigma'$.

(39% inside $\pi\cdot\sigma\cdot\sigma'$
99% inside $\pi\cdot 3\sigma\cdot 3\sigma'$)

6. Binomial distribution

This distribution has some interesting properties:

- the general case is characterised by a single parameter m > 0 and includes the waterbag model (uniform density), the parabolic distribution etc. The Kapchinsky-Vladimirsky-distribution[12] (KV)and the Gaussian distribution are the limiting cases $m \to 0$ and $m \to \infty$ respectively.
- for a finite m the distribution has a finite range, i.e. there are no particles outside a given limiting ellipse. The advantage against a truncated Gaussian e.g. is, that for $m \geq 2$ the beam profile has a continous derivative.

- projection of a binomial distribution in N-dimensional space into the (N-1)-dimensional space increases the parameter m by 0.5. This means that the distribution remains binomial under projection. In principle a specific distribution can be obtained by a series of projections of an inital K-V-distribution (for m = half integer).

To simplify the calculations we work with the dimensionless cartesian coordinates (u,v) or the polar coordinates (a, ϕ):

$$u = a\cos\phi \;,\quad v = a\sin\phi \qquad a^2 = u^2 + v^2 \tag{A.10}$$

The phase space density $\rho(u,v)$ is assumed to be rotationally symmetric and to have a binomial form

$$\boxed{\begin{aligned} \rho(u,v) &= \frac{m}{\pi}(1-a^2)^{m-1} && \text{for } a \leqslant 1 \\ &= 0 && \text{for } a \geqslant 1 \end{aligned}} \tag{A.11}$$

The distribution is normalized:

$$\iint \rho(u,v)\,du\,dv = \int_0^{2\pi}\int_0^1 \rho(a)\,a\,da\,d\phi = 1 \tag{A.12}$$

This phase space density $\rho(u,v)$ is analogous to the form

$$\begin{aligned} f(H) &= n\,B\,(H_o-H)^{n-1} && \text{for } H \leqslant H_o \\ &= 0 && \text{for } H > H_o \end{aligned} \tag{A.13}$$

used by W. Lysenko[13] from LAMPF for space charge calculations in Linacs. H is the Hamiltonian which for linear forces is quadratic in all variables and has thus the same form as a^2. The case $n \to 0$ gives $f(H) = \delta(H-H_0)$ and is called microcanonical or KV-distribution. The case $n \to \infty$ gives $f(H) = \exp(-H/H_0)$ and is called canonical or Maxwell distribution. We have introduced the parameter m for the two dimensional distribution in order to distinguish it from n in the Hamiltonian, which can be of higher dimension.

The connection between the dimensionless variables (u,v) and the real phase space variables (x,x') is given by

$$\left.\begin{aligned} x &= x_L \cdot u &&= x_L \, a \cos\phi \\ x' &= x'_L \, (u \sin\chi + v \cos\chi) &&= x'_L \, a \sin(\phi+\chi) \end{aligned}\right| \qquad (A.14)$$

where

$$\left.\begin{aligned} x_L &= \sqrt{\frac{m+1}{2}} \cdot x_m = \sqrt{2(m+1)}\,\sigma = \text{limiting amplitude} \\ x'_L &= \sqrt{\frac{m+1}{2}} \cdot \theta_m = \sqrt{2(m+1)}\,\sigma' = \text{limiting divergence} \end{aligned}\right| \qquad (A.15)$$

The RMS-ellipse is characterized by $x_m=2\sigma$, $\theta_m=2\sigma'$ and χ as given in (35). The <u>limiting</u> beam ellipse is characterized by x_L, x'_L and χ corresponding to a = 1. The <u>beam profile</u> is given by the projection of ρ onto u or x:

$$f(u) = \int_{-\infty}^{+\infty} \rho(u,v)dv = \frac{m}{\pi} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (1-u^2-v^2)^{m-1} dv \qquad (A.16)$$

with the substitution $v = \sqrt{1-u^2}\,\sin\alpha$ one obtains

$$f(u) = \frac{m}{\pi} I_{2m-1} \cdot (1-u^2)^{m-1/2} \qquad (A.17)$$

Thus f(u) has the same form as ρ(a) in (A.11) with the exponent increased by .5. The normalization constant I_K is given by

$$I_K = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^K\alpha d\alpha = \int_{-1}^{+1} (1-v^2)^{\frac{K-1}{2}} dv \qquad (A.18)$$

and obeys the recurrence relation:

$$I_K = \frac{2\pi}{K\, I_{K-1}} = \frac{K-1}{K} I_{K-2} \qquad (A.19)$$

The first few constants are

for K even:

$I_0 = \pi$

$I_2 = \pi \cdot 1/2$

$I_4 = \pi \cdot \frac{1\cdot 3}{2\cdot 4}$

⋮

$I_K = \pi \cdot \frac{1\cdot 3}{2\cdot 4} \cdots\cdot \frac{K-1}{K}$

for K odd:

$I_1 = 2$

$I_3 = 2\cdot 2/3$

$I_5 = 2\cdot \frac{2\cdot 4}{3\cdot 5}$

⋮

$I_K = 2\cdot \frac{2\cdot 4}{3\cdot 5} \cdots\cdot \frac{K-1}{K}$

Equation (A.19) allows to rewrite the profile f(u) in (A.17) as:

$$f(u) = \frac{1}{I_{2m}} (1-u^2)^{m-1/2} \quad \text{for } |u| \leq 1$$
$$= 0 \quad \text{for } |u| > 1 \qquad (A.20)$$

with the normalization $\int_{-\infty}^{+\infty} f(u)du = 1$

The profile f(u) is defined even for m = 0, although the original density ρ (u,v) in (A.11) is not defined for this case. The corresponding density $\rho_o(u,v)$ whose projection is f(u) for m = 0 is given by:

$$\rho_o(u,v) = \frac{1}{\pi}\,\delta(1-a^2) \qquad (A.21)$$

The variances σ^2 and σ'^2 for the general case are:

$$\sigma^2 \equiv \overline{x^2} = x_L{}^2 \int_{-\infty}^{\infty} u^2 f(u)\,du = \frac{x_L{}^2}{2(m+1)}$$

$$\sigma'^2 \equiv \overline{x'^2} = \frac{x_L'^2}{2(m+1)} \qquad (A.22)$$

This result has already been anticipated in (A.15) - Again we would like to know what fraction p of the beam is lying outside a certain ellipse given by a = const. in (A.14). We obtain

$$1-p = \int_0^{2\pi}\int_0^{a} \rho(a)\,a\,da\,d\phi = 1-(1-a^2)^m$$

$$\text{or } p = (1-a^2)^m \qquad (A.23)$$

Since the emittance area of this ellipse is

$$\pi\varepsilon_p = \pi\, x_L x_L'\, a^2 \cos\chi \qquad (A.24)$$

and the RMS-emittance area is

$$\pi\varepsilon = \pi x_m\,\theta_m \cos\chi = \pi\frac{2}{m+1}\,x_L x_L' \cos\chi \qquad (A.25)$$

we can write $\frac{\varepsilon_p}{\varepsilon} = a^2 \frac{m+1}{2}$ and rewrite (A.23) as:

$$p(\varepsilon_p) = \left(1 - \frac{2}{m+1}\frac{\varepsilon_p}{\varepsilon}\right)^m \qquad (A.26)$$

In the limit $m \to \infty$ we have

$$p(\varepsilon_p) \to \exp\left(-2\frac{\varepsilon_p}{\varepsilon}\right) \qquad (A.27)$$

which is the same value as for the Gaussian in (A.9). This indicates that for $m \to \infty$ the binomial distribution converges toward the 2-dimensional Gaussian with x_L and x'_L becoming infinite. In this case we have to take the standard deviations σ and σ' as normalization units for amplitude and divergence. If Γ is the full width half maximum value (=FWHM) of the binomial, then Γ/σ converges with increasing m towards the value $\sqrt{8 \ln 2}$, the same value as for a Gaussian. The advantage of the binomial against the Gaussian is that x and x' have upper bounds.

Fig. 5 shows 3 typical profiles for m = 1.5 (parabolic profile) m = 3 and m = ∞ (Gaussian), all having the same amplitude $x_m = 2\sigma$. The figure suggests a quick recipe to estimate the RMS-amplitude $x_m = 2\sigma$ by inspection:

If a beam profile looks approximately binomial (with m > 1.5) the amplitude where the intensity drops to the 15 % level is approximately x_m, rather independent of m.

Curves ln p versus $\varepsilon_p/\varepsilon$ constructed from (A.26) and (A.27) are shown in Fig. 6 for different values of m. It is very convenient to display the result of beam emittance measurement in such a plot, since a Gaussian distribution leads to a straight line. Binomial or other non Gaussian distributions can easily be identified in the beam tail region. Fig. 6 shows that the definition of a RMS-ellipse from the amplitudes $x_m = 2\sigma$ is a reasonable choice, since the fraction of particles $p(\varepsilon)$ outside the RMS-emittance $\pi\varepsilon$, given by (A.26) as

$$p(\varepsilon) = \left(\frac{m-1}{m+1}\right)^m \qquad (A.28)$$

is rather insensitive to m-values over 2 and approaches 13.5% for the Gaussian (see table 2). A closer analysis shows that the definition $x_m = 1.8\sigma$ leads to an emittance containing about 80% of the beam, even less dependent on m. But we will stick to the convential choice of $x_m=2\sigma$. Table 2 shows a summary of some parameters for different binomial phase space distributions $\rho(u,v)$ and their projected profiles f(u). It displays nicely the feature, that projection keeps the binomial form invariant. As an example how a given binomial distribution is obtained from projections we consider the case of a uniformly filled sphere in 6-dimensional phase space (x_1, ... x_6), which in turn is obtained from an 8-dimensional KV- or δ-distribution.

sphere: $$\sum_{i=1}^{6} x_i^2 \leqslant 1$$

$$\rho_6(x_1,\ldots,x_6) = \frac{6}{\pi^3}$$

$$\rho_5(x_1,\ldots,x_5) = \frac{12}{\pi^3}\,(1-x_1^2-\ldots-x_5^2)^{.5}$$

$$\rho_4(x_1,\ldots,x_4) = \frac{6}{\pi^2}\,(1-x_1^2-x_2^2-x_3^2-x_4^2)$$

$$\rho_3(x_1,x_2,x_3) = \frac{8}{\pi^2}\,(1-x_1^2-x_2^2-x_3^2)^{1.5}$$

$$\rho_2(x_1,x_2) = \frac{3}{\pi}\,(1-x_1^2-x_2^2)^2$$

$$\rho_1(x_1) = \frac{16}{5\pi}\,(1-x_1^2)^{2.5}$$

$$\rho_0 = 1$$

This case is identified with m = 3 from table 2 and is quite representative of typical beam profiles. The relation between a general binomial with parameter m and the dimension N of the uniform sphere is N=2m. The dimension of the corresponding K-V-distribution is N+2.

MONTE CARLO METHOD FOR GENERATING A BINOMIAL DISTRIBUTION

To generate a random distribution $\rho(x,x')$ obeying (A.11) one has to "invert" the binomial distribution. The key to this solution is (A.23) where we solve for a(1-p). The result is the following algorithm:

1) Specify the RMS-ellipse with x_m, θ_m and χ and choose the parameter m (>0) of the binomial $\rho(x,x')$.

2) Calculate the limiting amplitude and divergence x_L, x_L' :

$$x_L = \sqrt{\frac{m+1}{2}}\, x_m \quad , \quad x_L' = \sqrt{\frac{m+1}{2}}\, \theta_m$$

3) Choose random numbers S_1, S_2 (0 ÷ 1) and build

$$a = \sqrt{1 - S_1^{1/m}}$$

$$\alpha = 2\pi\, S_2$$

$$u = a \cos \alpha$$

$$v = a \sin \alpha$$

$$x = x_L\, u$$

$$x' = x_L' \,(u \sin \chi + v \cos \chi)$$

4) plot (x,x') and repeat 3) for next point

For a Gaussian ($m = \infty$) we replace the formula for a, x, x' by:

$$a = \frac{\sqrt{2}}{2} \sqrt{-\ln S_1} \quad , \quad (S_1 \neq 0)$$

$$x = x_m\, u$$

$$x' = \theta_m\, (u \sin \chi + v \cos \chi)$$

Fig. 7 shows plots for binomial distributions for the cases m = .5, 1, 3 and ∞, all having the same RMS-ellipse for 1000 particles.

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Summary of ellipse parameter

given / wanted	parametric $x_m,\ \theta_m,\ \chi$ $\varepsilon \equiv x_m\theta_m\cos\chi$ $r_{12} \equiv \sin\chi$	Courant-Snyder $\alpha,\ \beta,\ \gamma,\ \varepsilon$ $\beta\gamma - \alpha^2 \equiv 1$	Complex $R,\ S,\ \varepsilon$ $Y \equiv R+iS = \|Y\|\, e^{i\chi}$ $Y^{-1} \equiv U+iV$
x_m = max. ampl. θ_m = max. div. χ	x_m θ_m χ	$\sqrt{\varepsilon\beta}$ $\sqrt{\varepsilon\gamma}$ $-\tan^{-1}\alpha$	$\sqrt{\frac{\varepsilon}{R}}\sqrt{R^2+S^2} = \theta_m\|Y\|$ $\sqrt{\frac{\varepsilon}{R}}$ $\tan^{-1}\frac{S}{R} = \arg(Y)$
α β γ	$-\tan\chi$ $\frac{x_m}{\theta_m\cos\chi} = \frac{x_m^2}{\varepsilon}$ $\frac{\theta_m}{x_m\cos\chi} = \frac{\theta_m^2}{\varepsilon}$	α β γ	$-\frac{S}{R}$ $R+\frac{S^2}{R} = \frac{1}{R}\|Y\|^2$ $\frac{1}{R}$
R=ratio of main axis at waist S = distance from waist Y = R + iS	$\frac{x_m}{\theta_m}\cos\chi$ $\frac{x_m}{\theta_m}\sin\chi$ $\frac{x_m}{\theta_m}e^{i\chi}$	$\frac{1}{\gamma}$ $-\frac{\alpha}{\gamma}$ $\frac{1}{\gamma}(1-i\alpha)$	R S Y
x_W = waist amplitude Def. of waist divergent beam	$x_m\cos\chi$ $\chi = 0$ $0<\chi\leq 90^0$	$\sqrt{\frac{\varepsilon}{\gamma}}$ $\alpha=0$ $-\infty<\alpha<0$	$\sqrt{\varepsilon R}$ $S = 0$ $0<S<\infty$
ellipse equation	$x=x_m\cos\delta$ $x'=\Theta_m\sin(\delta+\chi)$ $(0\leq\delta\leq 2\pi)$	$\gamma x^2+2\alpha xx'+\beta x'^2 = \varepsilon$	

Table 1

TABLE 2: PROPERTIES (of) BINOMIAL PHASE SPACE DISTRIBUTION

m	phase space density $\rho(u,v)$ $(a^2 \equiv u^2+v^2 \leqslant 1)$	profile $f(u)=\int_{-\infty}^{+\infty}\rho(u,v)dv$ $(\lvert u\rvert \leqslant 1, \lvert x\rvert \leqslant x_L)$	profile form	$\frac{x_L}{x_m} \equiv \frac{x_L}{2\sigma}$ $(\sigma^2 \equiv \overline{x^2})$	total emittance	$\frac{\Gamma}{x_m} \equiv \frac{\Gamma}{2\sigma}$ $(\Gamma = \text{FWHM})$	$\frac{\Gamma}{x_L}$	$\frac{f(x_m)}{f_{max}}$	p=fraction of beam outside RMS-ellipse	q=fraction of beam outside ampl. $\pm x_m$
0(KV)	$\frac{1}{\pi}\delta(1-a^2)$	$\frac{1}{\pi}(1-u^2)^{-.5}$		.707		(1.414)	(2)	-	-	-
.5	$\frac{1}{2\pi}(1-a^2)^{-.5}$	$\frac{1}{2}$		.866		1.732	2	-	-	-
1	$\frac{1}{\pi}$ (=waterbag)	$\frac{2}{\pi}(1-u^2)^{.5}$ semi-circle		1	$4\sigma\sigma'$	1.732	1.732	0	0	0
1.5	$\frac{3}{2\pi}(1-a^2)^{.5}$ (elliptic)	$\frac{3}{4}(1-u^2)$ parabola		1.118	$5\sigma\sigma'$	1.582	1.414	20%	8.9%	1.6%
2	$\frac{2}{\pi}(1-a^2)$	(0.849) ← $\frac{8}{3\pi}(1-u^2)^{1.5}$		1.225	$6\sigma\sigma'$	1.491	1,217	19.2%	11.1%	2.5%
2.5	$\frac{5}{2\pi}(1-a^2)^{1.5}$	(0.93) ← $\frac{15}{16}(1-u^2)^2$		1.323	$7\sigma\sigma'$	1.431	1.082	18.4%	12.0%	3.0%
3	$\frac{3}{\pi}(1-a^2)^2$	(1.019) ← $\frac{16}{5\pi}(1-u^2)^{2.5}$		1.414	$8\sigma\sigma'$	1.392	.984	17.7%	12.5%	3.3%
3.5	$\frac{7}{2\pi}(1-a^2)^{2.5}$	(1.094) ← $\frac{35}{32}(1-u^2)^3$		1.5	$9\sigma\sigma'$	1.361	.908	17.1%	12.8%	3.5%
⋮ 7	$\frac{7}{\pi}(1-a^2)^6$	$1.52\,(1-u^2)^{6.5}$		2	$16\sigma\sigma'$	1.27	.636	15.4%	13.3%	4.1%
m	$\frac{m}{\pi}(1-a^2)^{m-1}$	$\frac{1}{I_{2m}}(1-u^2)^{m-.5}$		$\sqrt{\frac{m+1}{2}}$		$\sqrt{2(1-c)(m+1)}$ $\left[c = 1/2^{\frac{1}{m-.5}}\right]$	$2\sqrt{1-c}$	$\left(\frac{m-1}{m+1}\right)^{m-.5}$	$\left(\frac{m-1}{m+1}\right)^{m}$	recursive formula
$m \to \infty$ (Gaussian)	$\frac{1}{2\pi\sigma\sigma'}\exp\left(-\frac{x^2}{2\sigma^2}-\frac{x'^2}{2\sigma'^2}\right)$	$\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$		∞		$1.177=\sqrt{2\ln 2}$	0	13.5%	13.5%	4.6%

$$Y = \left[1 - \frac{2}{m+1} \frac{x^2}{x_m^2} \right]^{m-\frac{1}{2}} \quad , x^2 \leq \frac{m+1}{2} x_m^2$$

$$= 0 \quad , x^2 \geq \frac{m+1}{2} x_m^2$$

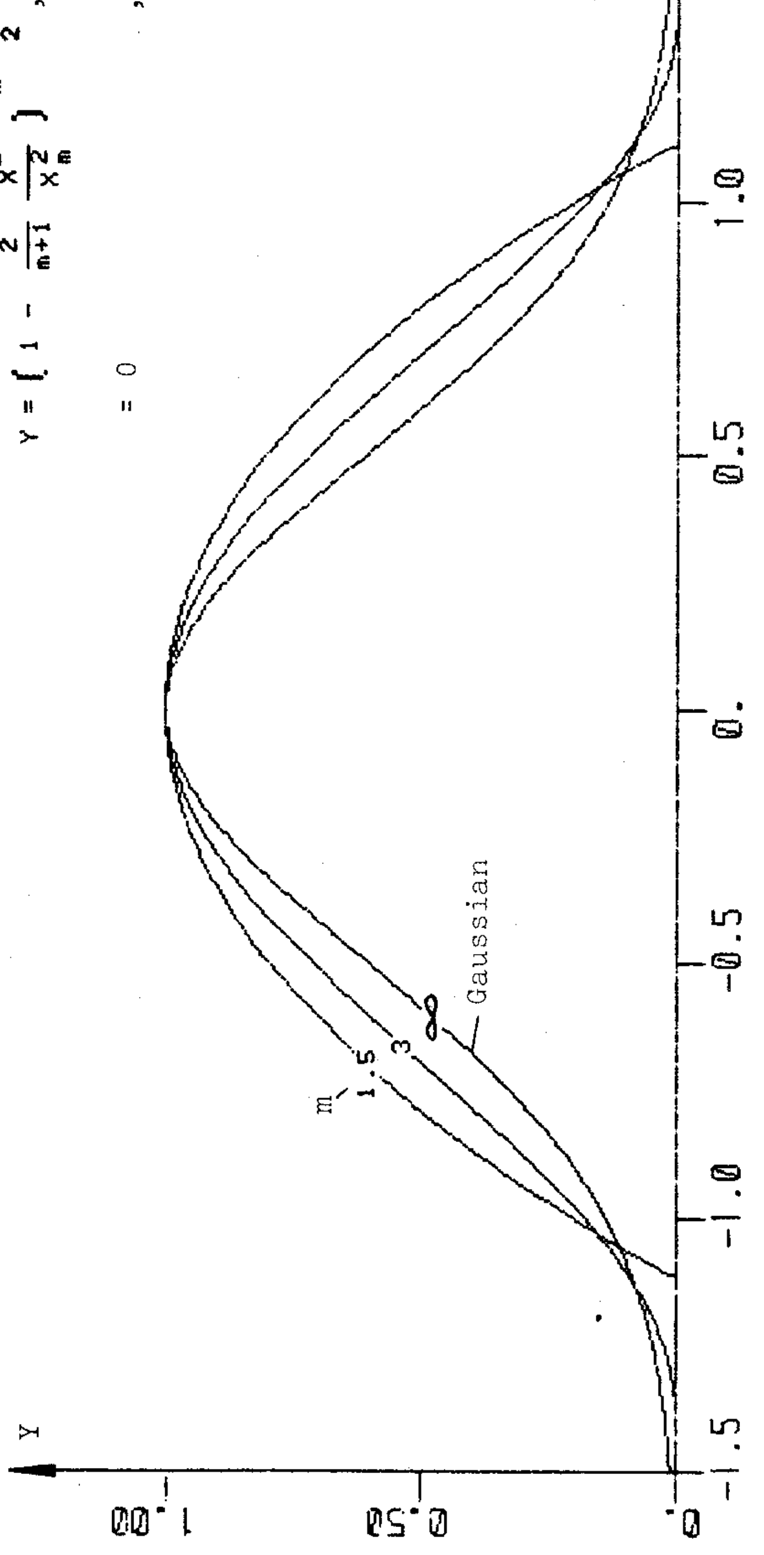

Figure 5: Binomial Profiles for m = 1.5, 3, ∞

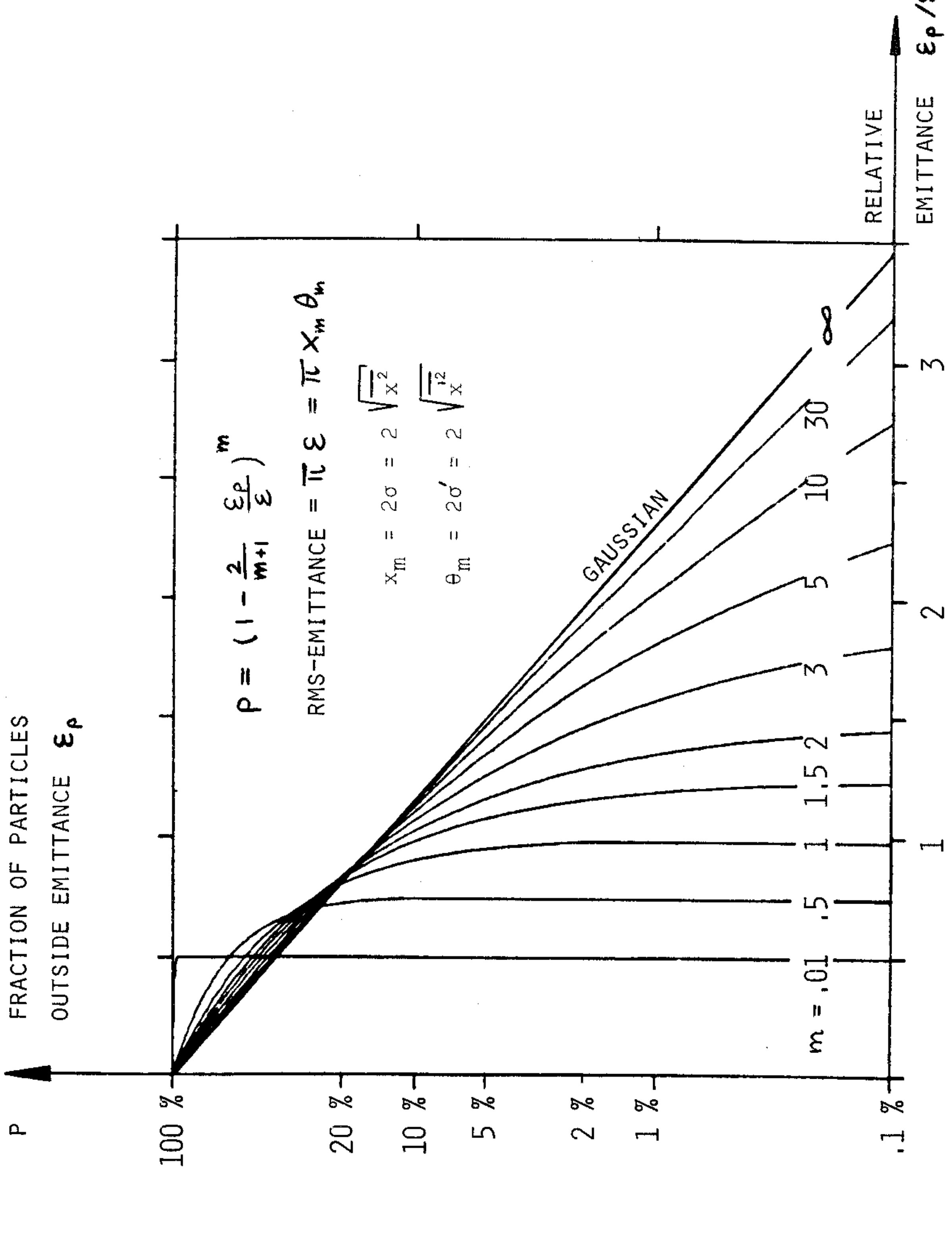
P
FRACTION OF PARTICLES
OUTSIDE EMITTANCE ε_p
100 %
20 %
10 %
5 %
2 %
1 %
.1 %
RELATIVE
EMITTANCE $\varepsilon_p/\varepsilon$
1
2
3
$P = \left(1 - \frac{2}{m+1} \frac{\varepsilon_p}{\varepsilon}\right)^m$
RMS-EMITTANCE = $\pi \varepsilon = \pi x_m \theta_m$
$x_m = 2\sigma = 2 \sqrt{\overline{x^2}}$
$\theta_m = 2\sigma' = 2 \sqrt{\overline{x'^2}}$
GAUSSIAN
m = .01
.5
1
1.5
2
3
5
10
30
∞

Figure 6:

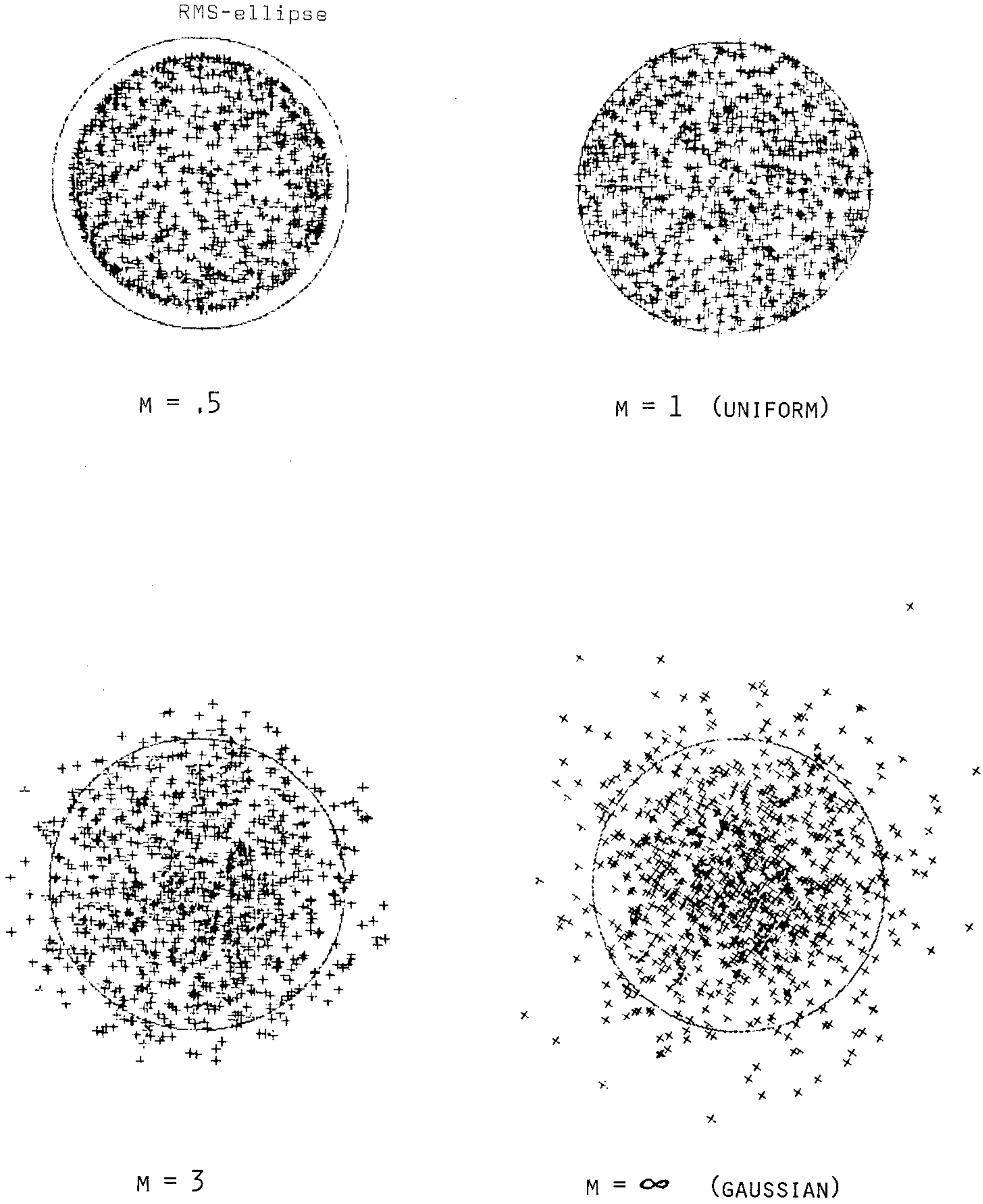

Figure 7: BINOMIAL PHASE SPACE DENSITIES