RMS ENVELOPE EQUATIONS WITH SPACE CHARGE[1]

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Summary

Envelope equations for a continuous beam with uniform charge density and elliptical cross-section were first derived by Kapchinsky and Vladimirsky[2](K-V). In fact, the K-V equations are not restricted to uniformly charged beams, but are equally valid for any charge distribution with elliptical symmetry, provided the beam boundary and emittance are defined by rms (root-mean-square) values. This results because (i) the second moments of any particle distribution depend only on the linear part of the force (determined by least squares method), while (ii) this linear part of the force in turn depends only on the second moments of the distribution. This is also true in practice for three-dimensional bunched beams with ellipsoidal symmetry, and allows the formulation of envelope equations that include the effect of space charge on bunch length and energy spread.

The utility of this rms approach was first demonstrated by Lapostolle[3] for stationary distributions. Subsequently, Gluckstern[4] proved that the rms version of the K-V equations remain valid for all continuous beams with axial symmetry. In this report these results are extended to continuous beams with elliptical symmetry as well as to bunched beams with ellipsoidal form, and also to one-dimensional motion.

Moment equations

Consider an ensemble of particles that obey the single-particle equations

$$\dot{x} = p \qquad \dot{p} = F(x,t) , \tag{1}$$

where $F(x,t)$ includes both the external force and the self-force, $F = F_e + F_s$. Averaging (1) over an arbitrary particle distribution $f(x,p,t)$, we obtain

$$\dot{\overline{x}} = \overline{p} \qquad \dot{\overline{p}} = \overline{F} = \overline{F_e} , \tag{2}$$

where the last equation follows because $\overline{F_s} = 0$ by Newton's third law. (We neglect the small magnetic self-forces due to internal motion.) If $F_e(x,t)$ is non-linear in x, the second equation of (2) involves the higher moments $\overline{x^n}$ of the distribution. However, for linear external forces, $F_e = -K(t)x$, equations (2) involve only the first moments $\overline{x}$ and $\overline{p}$, and therefore the centre-of-mass motion depends only on the external force,

$$\ddot{\overline{x}} + K(t)\overline{x} = 0 , \tag{3}$$

and not on the detailed form of the distribution. In the remainder of this paper we consider only linear external forces.

The second moments of $f(x,p,t)$ satisfy the equations

$$\begin{aligned} \dot{\overline{x^2}} &= 2\,\overline{x\dot{x}} = 2\,\overline{xp} \\ \dot{\overline{xp}} &= \overline{\dot{x}p} + \overline{x\dot{p}} = \overline{p^2} - K(t)\overline{x^2} + \overline{xF_s} \\ \dot{\overline{p^2}} &= 2\,\overline{p\dot{p}} = -2K(t)\overline{xp} + 2\,\overline{pF_s} , \end{aligned} \tag{4}$$

where the terms $\overline{xF_s}$ and $\overline{pF_s}$ are usually functions of the higher moments $\overline{x^n}$ and $\overline{x^np}$. This is a general feature of moment equations, namely the equation for each moment involves the higher moments in an endless hierarchy. However, if the self-force is derived from the free-space Poisson equation, $\overline{xF_s}$ depends mainly on the second moments and very little, if at all, on the higher moments. This will be demonstrated in the following sections. The remaining term $\overline{pF_s}$ is associated with emittance growth; we will avoid considering it by assuming that the rms emittance

$$E = \sqrt{\overline{x^2}\;\overline{p^2} - \overline{xp}^2} \tag{5}$$

is either constant, or that its time dependence is known *a priori*. Then $\overline{p^2}$ is given in terms of $\overline{x^2}$, $\overline{xp}$, and $E(t)$ by (5), and the first two equations of (4) form a closed set. They can be combined to give the K-V type equation:

$$\ddot{\tilde{x}} + K(t)\tilde{x} - \frac{E^2}{\tilde{x}^3} - \frac{\overline{xF_s}}{\tilde{x}} = 0 , \tag{6}$$

where $\tilde{x}$ is the rms value, $\tilde{x} = \sqrt{\overline{x^2}}$.

The space-charge term in this equation has an interesting interpretation. If we define the linear part of the force $F_s(x,t)$ as $\varepsilon(t)x$, where $\varepsilon(t)$ is determined by minimizing the difference

$$D = \int \left[\varepsilon(t)x - F_s(x,t)\right]^2 n(x,t)\, dx \tag{7}$$

for a fixed t, where $n(x,t) = \int f(x,p,t)\, dp$, then

$$\varepsilon(t)x = \frac{\overline{xF_s}}{\tilde{x}^2}\, x . \tag{8}$$

In other words, the rms envelope equation depends only on the linear part of the forces, determined by least squares method.

It is convenient to put equation (4) into matrix form. The assumption of constant rms emittance is equivalent to setting $\overline{pF_s} = \varepsilon(t)\overline{xp}$. Then equation (4) has the form

$$\dot{\sigma} = F\sigma + \sigma F^T \tag{9}$$

where σ is the covariance matrix

$$\sigma = \begin{bmatrix} \overline{x^2} & \overline{xp} \\ \overline{xp} & \overline{p^2} \end{bmatrix} \tag{10}$$

and F is

$$F = \begin{bmatrix} 0 & 1 \\ -K(t) + \varepsilon(t) & 0 \end{bmatrix} . \tag{11}$$

Equation (9) is equivalent to $\sigma(t + dt) = M\sigma(t)M^T$ where M is the infinitesimal transfer matrix $M(t + dt, t) = I + F(t)\, dt$.

This procedure is easily extended to two and three dimensions. For three dimensions, the 6 × 6 correlation matrix includes cross-correlation terms such as $\overline{xy}$, $\overline{xy'}$, ..., while the 6 × 6 force matrix F may include linear coupling terms from both space-charge and external forces. The three-dimensional equivalent of (9) has been incorporated into program TRANSPORT[5] to investigate both longitudinal and transverse space-charge effects in transfer lines[6]. In many cases the external forces will not involve coupling and the cross-correlation terms between the different directions will be zero or close to zero. In this case the envelope equations reduce to the K-V form (6) for each direction.

One-dimensional envelope equations

For a beam in free space that is very long in the z-direction and very wide in the y-direction, only the x-component of the self-force is important, and this is obtained from the Poisson equation

$$\frac{\partial \varepsilon}{\partial x} = 4\pi e n(x,t) \ . \tag{12}$$

The envelope equation is

$$\ddot{\tilde{x}} + K(t)\tilde{x} - \frac{E^2}{\tilde{x}^3} - \frac{e}{mN}\frac{\overline{x\varepsilon}}{\tilde{x}} = 0 \ , \tag{13}$$

where N is the number of particles per unit area in Δy Δz. This equation can be written as

$$\ddot{\tilde{x}} + K(t)\tilde{x} - \frac{E^2}{\tilde{x}^3} - \frac{2\pi e^2 N}{m}\lambda_1 = 0 \ , \tag{14}$$

where λ_1 is the dimensionless parameter

$$\lambda_1 = \frac{2\int_{-\infty}^{\infty} x h(x)\,dx \int_0^x h(x')\,dx'}{\left[\int_{-\infty}^{\infty} x^2 h(x)\,dx\right]^{1/2}} \tag{15}$$

and where $h(x) = (1/N)n(x)$ specifies the distribution. For the four distributions

a) uniform, $h(x) = \frac{1}{2}$ for $x \leq 1$; $= 0$ for $x > 1$

b) parabolic $h(x) = \frac{3}{4}(1 - x^2)$ for $x \leq 1$; $= 0$ for $x > 1$

c) gaussian, $h(x) = \frac{1}{2}\, e^{-x^2/2}$

d) hollow, $h(x) = \frac{1}{2}\, x^2 e^{-x^2/2}$,

the values of λ_1 are given in Table 1.

Table 1

Distribution	$\sqrt{3}\,\lambda_1$	$\frac{10\sqrt{5}}{3}\lambda_2$	$5\sqrt{5}\,\lambda_3$
uniform	1	1.08	1
parabolic	0.996	1	1.01
gaussian	0.977	1.05	1.05
hollow	0.987	1.37	1.02

Thus, for the range of distributions likely to be encountered in practice, the variation in λ_1 is negligible and the rms envelope motion will be accurately described by Eq. (14) with constant λ_1, for example $\lambda_1 = 1/\sqrt{3}$.

A second type of one-dimensional envelope equation arises in the study of longitudinal oscillations of a bunched beam inside a conducting pipe[7]. The longitudinal self-field is determined by

$$\varepsilon(z,t) = -eg\,\frac{\partial n(z,t)}{\partial z} \ , \tag{16}$$

where g = 1 + 2 ln (pipe radius/beam radius), and the corresponding envelope equation is

$$\ddot{\tilde{z}} + K(t)\tilde{z} - \frac{E^2}{\tilde{z}^3} - \frac{ge^2N}{m}\frac{\lambda_2}{\tilde{z}^2} = 0 \ , \tag{17}$$

where N is the number of particles per bunch and

$$\lambda_2 = \frac{1}{2}\left[\int_{-\infty}^{\infty} z^2 h(z)\,dz\right]^{1/2} \int_{-\infty}^{\infty} h^2(z)\,dz \tag{18}$$

with values of λ_2 listed in Table 1. For this case of a shielded electric field, the envelope equation does depend on the type of distribution. However, if the form of the distribution varies only slightly during its evolution, for example remains within the range uniform-parabolic-Gaussian, then the envelope equation (17) can be used with confidence.

Envelope equations for continuous beams

In the absence of cross-correlations and coupling terms, the envelope equations have the form (13) where the space-charge terms involve the average $\overline{x\varepsilon_x}$ and $\overline{y\varepsilon_y}$. These averages will depend only on the second moments $\tilde{x}$ and $\tilde{y}$ and not on the higher moments provided the charge distribution has the elliptical symmetry

$$n(x,y,t) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\ ,\ t\right) \ . \tag{19}$$

In this case the solution to Poisson's equation is

$$\varepsilon_x = 2\pi eabx \int_0^{\infty} \frac{n(T)\,ds}{(a^2+s)^{3/2}(b^2+s)^{1/2}} \ , \tag{20}$$

where

$$T = \frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} \ , \tag{21}$$

with a similar expression for ε_y. The term $\overline{x\varepsilon_x}$ is therefore

$$\overline{x\varepsilon_x} = 2\pi eab \int_0^{\infty} ds \int_{-\infty}^{\infty} \frac{x^2\,dx}{(a^2+s)^{3/2}} \int_{-\infty}^{\infty} \frac{dy}{(b^2+s)^{1/2}}\, n(T)\, n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \tag{22}$$

which suggests the change of variables

$$r\cos\theta = \frac{x}{\sqrt{a^2+s}}\ , \quad r\sin\theta = \frac{y}{\sqrt{b^2+s}} \ . \tag{23}$$

With the new variables, the integration over θ can be performed giving

$$\overline{x\varepsilon_x} = \frac{4\pi e a^3 b^2}{a+b} \int_0^{\infty} n(r^2) 2\pi r\,dr \int_r^{\infty} n(r'^2) 2\pi r'\,dr' \ . \tag{24}$$

The remaining integrals can be evaluated with the help of the definition

$$N = \iint_{-\infty}^{\infty} n(x,y)\,dx\,dy = ab\int_0^{\infty} n(r^2) 2\pi r\,dr \ , \tag{25}$$

where N is the number of particle per unit length. Then

$$Q(r) = ab\int_0^r n(r'^2) 2\pi r'\,dr' \tag{26}$$

is the number of particles within radius r, and Eq. (24) becomes

$$\overline{x\varepsilon_x} = \frac{2ea}{a+b}\int_0^\infty \frac{dQ}{dr'}\left[N - Q(r')\right] dr' \ , \qquad (27)$$

which is easily integrated,

$$\overline{x\varepsilon_x} = \frac{eN^2 a}{a+b} = \frac{eN^2\tilde{x}}{\tilde{x}+\tilde{y}} \ . \qquad (28)$$

Using this and the expression for $\overline{y\varepsilon_y}$, we obtain the envelope equations

$$\ddot{\tilde{x}} + K_x(t)\tilde{x} - \frac{E_x^{\,2}}{\tilde{x}^3} - \frac{e^2N}{m}\frac{1}{\tilde{x}+\tilde{y}} = 0$$

$$\ddot{\tilde{y}} + K_y(t)\tilde{y} - \frac{E_y^{\,2}}{\tilde{y}^3} - \frac{e^2N}{m}\frac{1}{\tilde{x}+\tilde{y}} = 0 \ . \qquad (29)$$

These equations are identical to the K-V equations if the rms values $\tilde{x}$, E_x, $\tilde{y}$, E_y are replaced by the physical boundary for a uniform distribution, namely $X = 2\,\tilde{x}$, ... However, they are not restricted to the K-V distribution but are valid for any distribution with the elliptical symmetry (19).

Envelope equations for bunched beams

The procedure in two-dimensions can be repeated for bunched beams with the ellipsoidal symmetry

$$n(x,y,z,t) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \ , \ t\right) \ . \qquad (30)$$

The electric field is[8]

$$\varepsilon_x = 2\pi eabcx\int_0^\infty \frac{n(T)\ ds}{(a^2+s)^{3/2}(b^2+s)^{1/2}(c^2+s)^{1/2}} \ , \qquad (31)$$

where

$$T = \frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s} \ , \qquad (32)$$

and with analogous expressions for ε_y and ε_z. The term $\overline{x\varepsilon_x}$ can be reduced to the form

$$\overline{x\varepsilon_x} = \frac{eN^2\lambda_3}{\tilde{x}}\, g_x\left(\frac{b}{a}, \frac{c}{a}\right) \ , \qquad (33)$$

where N is the number of particles per bunch and

$$g_x = \frac{3}{2}\int_0^\infty \frac{ds}{(1+s)^{3/2}\left(\frac{b^2}{a^2}+s\right)^{1/2}\left(\frac{c^2}{a^2}+s\right)^{1/2}} \ . \qquad (34)$$

The integral in (34) can be expressed in terms of elliptic integrals of the second kind, but direct numerical evaluation with the Gaussian integration method is easier and also quick and accurate. The complete envelope equation for $\tilde{x}$ is

$$\ddot{\tilde{x}} + K_x(t)\tilde{x} - \frac{E_x^{\,2}}{\tilde{x}^3} - \frac{e^2N\lambda_3}{m\tilde{x}^2}\, g_x\left(\frac{\tilde{y}}{\tilde{x}} \ , \ \frac{\tilde{z}}{\tilde{x}}\right) = 0 \ , \qquad (35)$$

where

$$\lambda_3 = \frac{1}{3\sqrt{3}}\left[\int_0^\infty h(r^2)r^4\ dr\right]^{1/2}\int_0^\infty h(r^2)r^2\ dr\int_r^\infty h(\rho^2)\rho\ d\rho \qquad (36)$$

with the normalization

$$\int_0^\infty h(r^2)r^2\ dr = 1 \ . \qquad (37)$$

The parameter λ_3 depends only weakly on the type of distribution as shown in Table 1. Thus for practical distributions, the dependence of the envelope equations on the type of distribution can be neglected. The same statement also applies if cross-correlations or linear external coupling forces are present; in this case the more general matrix form (9) of the rms equations can be used.

Conclusion

A rather surprising and useful result has been found for beams in free space, namely that the linear part of the self-field depends mainly on the rms size of the distribution and only very weakly on its exact form. Using this result, envelope equations for the rms beam size have been derived that are exact for continuous beams of elliptical symmetry, and in practice also valid for bunched beams of ellipsoidal form. The main restriction in applying these equations is that the time dependence of the rms emittance must be known *a priori*.

Possible uses of the equations include the specification of stationary or matched states in the presence of space charge. For example, the periodic solution of Eq. (35) for alternating-gradient structures, including radio frequency cavities, specifies the matched beam size (both longitudinal and transverse) as a function of rms emittances and intensity. The largest matched size attainable without exceeding aperture limits or bucket size determines a space-charge limit. For a beam matched in this way, envelope oscillations about the periodic solution are suppressed, although higher modes of oscillations (sextupole, octupole, etc.) may occur. Suppression of the higher modes will require constraints, as yet undetermined, on the higher moments of the distribution. Another use is the design of low-energy beam transfer lines.

References

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